ELEMENTARY

TEXT-BOOK ON PHYSICS

BY

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PREFACE.

In the preparation of this work, we have had in view the presentation, to our own classes, of the Science of Physics in its present condition. We have undertaken the task because we have been unable to find among existing text-books any one suited to our needs.

The demonstrations given presuppose a knowledge of mathematics through plane trigonometry. In two or three cases, to make a subject complete, results have been derived requiring a knowledge of analytical geometry for their full comprehension. The discussions leading to these results may, however, be omitted without any breach of continuity of the work.

In the selection of the matter to be presented in this "First Part" of the work, we have had in view the subjects, Electricity and Magnetism, and Acoustics and Optics, which are to follow; and we have endeavored to give such a treatment of the laws of motion, and of the action of force, as would enable the student to master easily the mechanical problems presented in the action of electrical forces, and in the study of the vibratory motions constituting sound and light.

In the treatment of the subject, we have endeavored to be concise and exact, and we have tried to present the matter in the simplest form that a rigid treatment would admit of.

We are well aware that to master some of the subjects presented, will require hard work on the part of the student; but we have not for that reason deemed it advisable to omit those subjects, or to confine ourselves to a mere statement of results reached.

In the preparation of the work, we have made free use of all accessible sources of information. We wish especially to acknowledge our indebtedness to the works of Thomson, Tait, Maxwell, Jamin, Violle, Müller, and Stewart.

Part II., which will complete the work, is in progress, and will be issued as soon as circumstances will admit.

WILLIAM A. ANTHONY.
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BOOK I.

MECHANICS.

INTRODUCTION.

1. ***Divisions of Natural Science.*** — Every thing which can affect our senses, we call *matter*. Any limited portion of matter, however great or small, is called a *body*. All bodies, together with their unceasing changes, constitute **Nature.**

Natural Science makes us acquainted with the properties of bodies, and with the changes, or *phenomena*, which result from their mutual actions. It is therefore conveniently divided into two principal sections, — **Natural History** and **Natural Philosophy.**

The former describes natural objects, classifies them according to their resemblances, and, by the aid of Natural Philosophy, points out the laws of their production and development. The latter is concerned with the laws which are exhibited in the mutual action of bodies on each other.

These mutual actions are of two kinds: (*a*) those which leave the essential properties of bodies unaltered, and (*b*) those which effect a complete change of properties, resulting in loss of identity. Changes of the first kind are called *physical* changes: those of the second kind are called *chemical* changes. Natural Philosophy has, therefore, two subdivisions, — **Physics** and **Chemistry.**

Physics deals with all those phenomena of matter that are not directly related to chemical changes. **Astronomy** is thus a branch of Physics, yet it is usually excluded from works like the present on account of its special character.

It is not possible, however, to draw sharp lines of demarkation between the various departments of Natural Science, for the successful pursuit of knowledge in any one of them requires some acquaintance with the others.

2. ***Methods.*** — The ultimate basis of all our knowledge of nature is experience, — experience resulting from the action of bodies on our senses, and the consequent affections of our minds.

When a natural phenomenon arrests our attention, we call the result an *observation*. Simple observations of natural phenomena, only in rare instances can lead to such complete knowledge as will suffice for a full understanding of them. An observation is the more complete, the more fully we apprehend the attending circumstances. We are generally not certain that all the circumstances which we note are *conditions* on which the phenomenon, in a given case, depends. In such cases we modify or suppress one of the circumstances, and observe the effect on the phenomenon. If we find a corresponding modification or failure with respect to the phenomenon, we conclude that the circumstance, so modified, is a condition. We may proceed in the same way with each of the remaining circumstances; leaving all unchanged, except the single one purposely modified, at each trial, and always observing the effect of the modification. We thus determine

the conditions on which the phenomenon depends. In other words, we bring *experiment* to our aid in distinguishing between the real conditions on which a phenomenon depends, and the merely accidental circumstances which may attend it.

But this is not the only use of experiment. By its aid we may frequently modify some of the conditions, known to be conditions, in such ways that the phenomenon is not arrested, but so altered in the rate with which its details pass before us that they may be easily observed. Again: experiment often leads to new phenomena, and to a knowledge of activities before unobserved. Indeed, by far the greater part of our knowledge of natural phenomena has been acquired by means of experiment. To be of value, experiments must be conducted with system, and so as to trace out the whole course of the phenomenon.

Having acquired our facts by observation and experiment, we seek to find out how they are related; that is, to discover the *laws* which connect them. The process of reasoning by which we discover such laws is called *induction.* As we can seldom be sure that we have apprehended all the related facts, it is clear that our inductions must generally be incomplete. Hence it follows, that conclusions reached in this way are at best only probable; yet their probability becomes very great when we can discover no outstanding fact, and especially so when, regarded provisionally as true, they enable us to foresee what will occur in cases before unknown.

In conducting our experiments, and our reasonings likewise, we are often guided by suppositions, suggested by previous experience. If the course of our experi-

ment be in accordance with our supposition, there is, so far, a presumption in its favor. So, too, in reference to our reasonings: if all our facts are seen to be consistent with some supposition not unlikely in itself, we say it thereby becomes probable. The term *hypothesis* is usually employed instead of supposition.

Concerning the ultimate modes of existence or action, we know nothing whatever; hence, a law of nature cannot be demonstrated in the sense that a mathematical truth is demonstrated. Yet so great is the constancy of uniform sequence with which phenomena occur in accordance with the laws which we discover, that we have no doubt respecting their validity.

When we would refer a series of ascertained laws to some common agency, we employ the term *theory*. Thus we find in the "wave theory" of light, based on the hypothesis of a universal ether of extreme elasticity, satisfactory explanations of the laws of reflection, refraction, diffraction, polarization, etc.

3. **Measurements.** — All the phenomena of nature occur in matter, and are presented to us in time and space.

Time and space are fundamental conceptions: they do not admit of definition. Matter is equally indefinable: its distinctive characteristic is its persistence in whatever state of rest or motion it may happen to have, and the resistance which it offers to any attempt to change that state. This property is called *inertia*. It must be carefully distinguished from inactivity.

Another essential property of matter is *impenetrability*, or the property of occupying space to the exclusion of other matter.

We are almost constantly obliged, in physical science, to measure the quantities with which we deal. We measure a quantity when we compare it with some standard of the same kind. A simple number expresses the result of the comparison.

If we adopt suitable units of length, time, and mass (or quantity of matter), we can express the measure of all other quantities in terms of these so-called *fundamental units.*

The actual operation of comparing a length with the standard is often difficult of direct accomplishment. This may arise from the minuteness of the object or distance to be measured, from the distant point (at which the measurement is to end) being inaccessible, or from the difficulty of accurately dividing our standard into very small fractional parts. In all such cases we have recourse to indirect methods, by which the difficulties are more or less completely obviated. It will readily occur to the student that the appliances of trigonometry are available.

It is important that we be able to estimate small fractions of our units. The **vernier** enables us to do this in the case of the unit of length with great convenience and accuracy. It consists of an accessory piece, fitted to slide on the principal scale, or arc, of the instrument to which it is applied. A portion of the accessory piece, equal to n divisions of the principal scale, is divided into n plus one or into n minus one divisions. In the former case the divisions are numbered in the same sense with those of the principal scale; in the latter they are numbered in the opposite sense. In either case we can measure a quantity accurately to the one nth part of

one of the primary divisions of the principal scale. The cut (Fig. 1) will make the construction and use of the vernier plain.

In Fig. 1, let 0, 1, 2, 3 . . . 10 be the divisions on the vernier; let 0, 1, 2, 3 . . . 10 be any set of consecutive divisions on the principal scale.

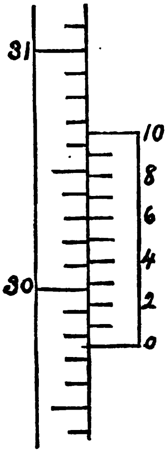

Fig. 1.

If we suppose the 0 of the vernier to be in coincidence with the limiting point of the magnitude to be measured, it is clear, that, from the position shown in the figure, we have 29.7, expressing that magnitude to the nearest tenth; and since the sixth division of the vernier coincides with a whole division of the principal scale, we have $\frac{6}{10}$ of $\frac{1}{10}$, or $\frac{6}{100}$ of a principal division to be added: hence the whole value is 29.76.

The **micrometer screw** is also much employed. It consists of a carefully cut screw, accurately fitting in a nut. The head of the screw carries a graduated circle, which can turn past a fixed point designated by a fiducial mark. This is frequently on a scale whose divisions are equal in magnitude to the pitch of the screw. These divisions will then show through how many revolutions the screw is turned in any given trial; while the divisions on the graduated circle will show the fractional part of a revolution, and consequently the fractional part of the pitch that must be added. If the screw be turned through n revolutions, as shown by the fiducial scale, and through an additional fraction, as shown by the divided

circle, it will pass through *n* times the pitch of the screw, and an additional fraction of the pitch determined by the ratio of the number of divisions read from o on the divided circle to the whole number, into which it is divided.

The **cathetometer** is used for measuring differences of level. A graduated scale is cut on an upright bar, which can turn about a vertical axis. Over this bar slide two accurately fitting pieces, one of which can be clamped to the bar at any point, and it can serve as the fixed bearing of a micrometer screw. The screw runs in a nut in the second piece, which has a vernier attached, and carries a horizontal telescope furnished with cross-hairs. The telescope having been made accurately horizontal by means of a delicate level, the cross-hairs are made to cover one of the points whose difference of level is sought, and the reading upon the scale is taken; the fixed piece is then unclamped, and the telescope raised or lowered until the second point is covered by the cross-hairs, and the scale reading is again taken. The difference of scale reading is the difference of level sought.

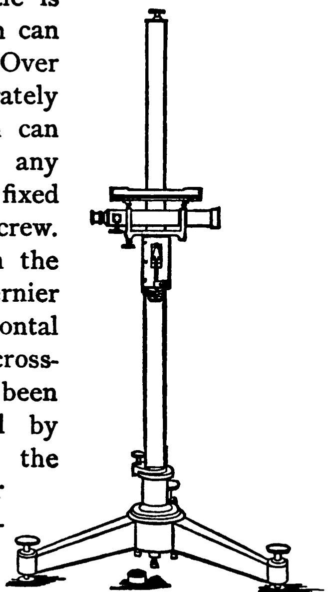

Fig. 2.

The **dividing engine** may be used for dividing scales or for comparing lengths. In its usual form it consists

essentially of a long micrometer screw, carrying a table, which slides, with a motion accurately parallel with itself, along fixed guides, resting on a firm support. To this table is fixed an apparatus for making successive cuts upon the object to be divided.

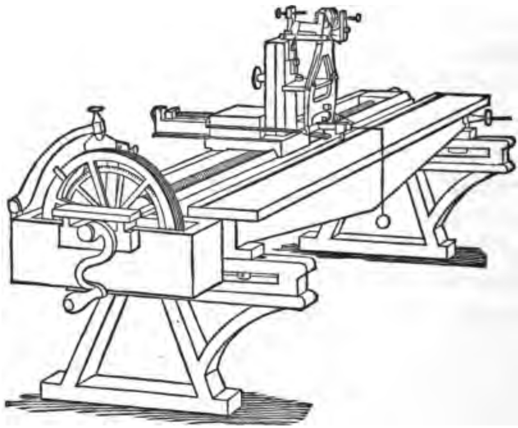

Fig. 3.

The object to be graduated is fastened to the fixed support. The table is carried along through any required distance determined by the screw, and the cuts are thus made at the proper intervals.

The same instrument, furnished with microscopes and accessories, may be employed for comparing lengths with a standard. It may then be called a **comparator.**

The **spherometer** is a special form of the micrometer screw. As its name implies, it is primarily used for measuring the curvature of spherical surfaces.

It consists of a screw with a large head, divided into a great number of parts, — say, five hundred, — turning

in a nut supported on three legs, whose pointed ends form the vertices of an equilateral triangle. The axis of revolution of the screw is perpendicular to the plane of the triangle, and passes through its centre. The screw ends in a point which may be brought into the same plane with the points of the legs. This is done by placing the legs on a truly plane surface, and turning the screw till its point is just in contact with the surface. The sense of touch will enable one to decide with great nicety when the screw is turned far enough. If, now, we note the reading of the divided scale, and also that of the divided head, and then raise the screw, by turning it backward, so that the given curved surface may exactly coincide with the four points, we can compute the radius of curvature from the difference of the two readings and the known length of the side of the triangle formed by the points of the tripod.

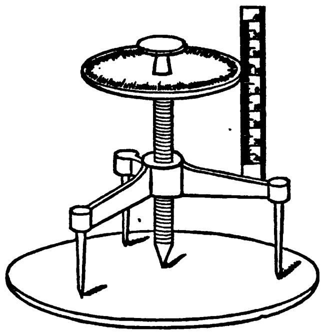

Fig. 4.

4. ***Units.*** — The **unit of length** is the **metre.** The standard is a certain piece of platinum, declared by legislative act to be the standard. It is preserved in the archives of France.

This standard was designed to be equal in length to one ten millionth of the earth's quadrant. Its multiples and submultiples are in accord with the decimal notation. The names of the former are derived from the Greek, the latter from the Latin. (The table will

exhibit these terms, and also the relation of the metric to other systems.)

It is often necessary to measure angles. For this purpose we employ the divided circle, on whose divisions are based the ordinary trigonometrical tables. For great accuracy in reading minute subdivisions, the vernier is attached. The telescope with cross-hairs is used, mounted on the axis of the circle, and turning with it, to secure alignment with the objects or points whose angular distance is to be found.

The **unit of time** is the mean time second, which is the $\frac{1}{86400}$ of a mean solar day. We employ the clock, regulated by the pendulum or the chronometer balance, to indicate seconds. This, while sufficient for ordinary use, must for accurate investigations be frequently corrected by astronomical observations.

Smaller intervals of time than the second are measured by causing some vibrating body, as a tuning-fork, to trace its path along some suitable surface, on which also the beginning and end of the event whose duration we wish to know are recorded. The number of vibrations traced while the event is occurring determines its duration in known parts of a second.

In estimating the duration of certain phenomena giving rise to light, the revolving mirror may be employed. By its use, with proper accessories, intervals as small as the $\frac{40}{1000000000}$ of a second have been estimated. — *Rood.*

The **unit of mass** may be any arbitrarily chosen piece of matter suitable for convenient reference. Those usually adopted are such as have been set apart by legislative enactment, and declared to be standards.

That one which is usually adopted in scientific work is the gram, which is the thousandth part of a standard kilogram, preserved in the archives of France. It is intended to be equal in mass to one cubic centimetre of water at its greatest density.

The names of the divisions and multiples are analogous to those of the metre. For the relation of the gram to the masses of other systems still used in commerce and in the arts, see table.

Masses are compared by means of the *balance*, the principles of whose construction will be discussed hereafter.

5. ***Dimensions of Units.*** — Any derived unit may be represented by the product of certain powers of the symbols representing the fundamental units of time, space, and mass. Thus, the unit of area varies as the square of the unit of length; hence its dimensional equation is area $= L^2$. In like manner, the dimensional equation for volume is vol. $= L^3$.

Any equation showing what powers of the fundamental units enter into the expression for the derived unit is called its dimensional equation.

CHAPTER I.

MECHANICS OF MASSES.

6. THE general subject of motion is usually divided, in extended treatises, into two topics, — **Kinematics** and **Dynamics.** In the first are developed, by purely mathematical methods, the laws of motion considered in the abstract, independent of any causes producing it and of any substance in which it inheres; in the second these mathematical relations are extended and applied, by the aid of a few inductions drawn from universal experience, to the explanation of the motions of bodies, and the discussion of the interactions which are the occasion of those motions.

For convenience merely, the subject of Dynamics is further divided into *Statics*, which treats of forces as maintaining bodies in equilibrium and at rest, and *Kinetics*, which treats of forces as setting bodies in motion.

In this treatise it has been found more convenient to make no formal distinction between the mathematical relations of motion and the application of those relations to the study of forces and the motions of bodies. The subject is so extensive that only those fundamental principles and results will be presented

which have direct application in subsequent parts of the work.

7. **Motion** is the change in position of a material particle. It is recognized by a change in the configuration of the system containing the displaced particle; that is, by a change in the relative positions of the particles making up the system. Any particle in the system may be taken as the fixed point of reference, and the motion of the others may be measured from it. Thus, for example, high-water mark on the shore may be taken as the fixed point in determining the rise and fall of the tides; or, the sun may be assumed to be at rest in computing the orbital motions of the planets. We can have no assurance that the particle which we assume as fixed is not really in motion as a part of some larger system; indeed, in almost every case we know that it is thus in motion. As it is impossible to conceive of a point in space recognizable as fixed and determined in position, our measurements of motion must always be relative.

One important limitation of this statement must be made: by proper experiments it is possible to determine the absolute angular motion of a body rotating about an axis.

8. **Path.** — The moving particle must always describe a continuous line or path. In all investigations the path may be represented by a diagram or model, or by reference to a set of assumed co-ordinates.

9. **Velocity.** — The velocity of a particle is its rate of motion. It may be either constant or variable. If constant, it is measured by the space traversed by the particle in a unit of time. If variable, it is measured

by the space which would be traversed by the particle in a unit of time if the velocity at the instant were to remain constant.

The practical unit of velocity is the velocity of a body moving uniformly through one centimetre in one second.

The velocity of a particle is given, therefore, by the quotient of the number of centimetres traversed divided by the number of seconds in which it is traversed.

The formula expressing uniform velocity is

$$v = \frac{s}{t}, \tag{1}$$

in which s represents space and t time.

The dimensions of velocity are LT^{-1}.

10. ***Momentum.***—The momentum of a body is its quantity of motion. This varies with its mass and its velocity jointly, and is measured by their product. Thus, for example, a body weighing ten grams, and having a velocity of ten centimetres, has the same momentum as a body weighing one gram, and having a velocity of one hundred centimetres. The practical unit of momentum is that of a gram of matter moving with the unit velocity. The formula is

$$mv, \tag{2}$$

where m represents mass.

The dimensions of momentum are MLT^{-1}.

11. ***Acceleration.***—When the velocity of a particle varies, its rate of change is called the acceleration of the particle. Acceleration is either positive or negative, according as the velocity increases or diminishes.

Acceleration, if constant, is measured by the change in the velocity of the particle in a unit of time: if variable, it is measured by what would be the change in the velocity if from that instant the acceleration were to remain constant.

The practical unit of acceleration is that of a particle whose velocity changes one unit of velocity in one second. Acceleration is therefore measured by the quotient of the number of units of change of velocity divided by the number of seconds during which the change has occurred.

The formula expressing a constant acceleration is

$$f = \frac{v - v_0}{t}, \tag{3}$$

in which v_0 and v represent the initial and final velocities respectively.

The dimensions of acceleration are LT^{-2}.

The space traversed by a point moving with a constant acceleration f, during a time t, is determined by considering, that, since the acceleration is constant, the average velocity $\frac{v + v_0}{2}$ for the time t, multiplied by t, will represent the space traversed: hence

$$s = \frac{v + v_0}{2} t, \tag{4}$$

or since $\frac{v}{2} = \frac{v_0 + ft}{2}$, we have, in another form,

$$s = v_0 t + \frac{1}{2} ft^2. \tag{4}$$

Multiplying equations (3) and (4), we obtain

$$v^2 = v_0^2 + 2fs. \tag{5}$$

When the point starts from rest, $v_0 = 0$, and equations (3), (4), and (5) become $v = ft$, $s = \frac{1}{2} ft^2$, and $v^2 = 2fs$.

Formula (4) may also be obtained by a geometrical construction.

At the extremities of a line AB (Fig. 5), equal in length to t, erect perpendiculars AC and BD, proportional to the initial and final velocities of the moving-point. For any interval of time Aa so short that the velocity during it may be considered constant, the space described is represented by the rectangle Ca, and the space described in the whole time t, by a point moving with a velocity increasing by successive increments, is represented by a series of rectangles, eb, fc, gd, etc., described on ab, bc, cd, etc. If ab, bc . . . are diminished indefinitely, the sum of the areas of the rectangles can be made to approach as nearly as we please the area of the quadrilateral $ABCD$. This area, therefore, represents the space traversed by the point, having the initial velocity v_0, and moving with the acceleration f, through the time t. But $ABCD$ is equal to $AC \cdot t + (BD - AC) t \div 2$; whence

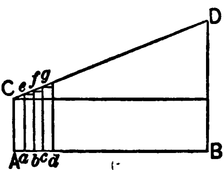

Fig. 5.

$$s = v_0 t + \frac{1}{2} ft^2. \qquad (4)$$

12. ***Composition and Resolution of Motions, Velocities, and Accelerations.*** — If a point a_1 move with a constant velocity relative to another point a_2, and this point a_2 move with a constant velocity relative to a third point a_3, then the motion, in any fixed time, of a_1 relative to a_3 may be readily found.

Represent the motion, in a fixed time, of a_1 relative to a_2 (Fig. 6) by the line v_2, and of a_2 relative to a_3 by the line v_3. Now, it is plain that the motions v_2 and v_3, whether acting successively or simultaneously, will bring the point a_1 to B; and also that if any portions of these motions Ab and bc, occurring in any small portion of time, be taken, they will, because the velocities of a_1 and a_2 are constant or proportional to v_2 and v_3, bring the point a_1 to some point c lying on the line joining A and B. Therefore the diagonal AB of the parallelogram having the sides v_2 and v_3 fully represents the motion of a_1 relative to a_3.

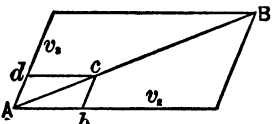

Fig. 6.

The line AB is called the resultant, of which the two lines v_2 and v_3 are the components.

This proposition may now be stated generally. The resultant of any two simultaneous motions, represented by two lines drawn from the point of reference, is found by completing the parallelogram of which those lines are sides, when the diagonal drawn from the point of reference represents the resultant motion.

The resultant of any number of motions may be found by obtaining the resultant of any two of the given components, by means of the parallelogram as before shown, using this resultant in combination with another component to obtain a new resultant, and proceeding in this way till all the components have been used.

Or, represent the given motion by one side of a polygon. Its components may be represented by the remaining sides. And conversely, if it be required to

find the resultant of a number of components, they may be laid off as the consecutive sides of a polygon when the line required to complete the polygon is the resultant sought.

Motions are usually resolved along three rectangular axes by means of the trigonometrical functions. Thus, if a be the line representing the motion, and θ, ϕ, and ψ the angles which it makes with the three axes, the components along those axes are $a \cos \theta$, $a \cos \phi$, and $a \cos \psi$.

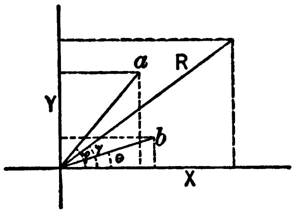

Fig. 7.

Two motions may be compounded in this way, as follows: For convenience, let us assume the co-ordinate axes in the same plane as the lines representing the motions. Then (Fig. 7) $a \cos \phi$, $b \cos \theta$, $a \sin \phi$, $b \sin \theta$, are the resolved components of a and b along the axes.

Let $a \cos \phi + b \cos \theta = X$, and $a \sin \phi + b \sin \theta = Y$; then the diagonal of the rectangle, of which X and Y are sides, is $R = (X^2 + Y^2)^{\frac{1}{2}}$; or, since the angle between the resultant and the axis of X is known by $Y = X \tan \psi$, it follows that $R = \dfrac{X}{\cos \psi}$ or $\dfrac{Y}{\sin \psi}$. It is evident that this process may be extended to any number of components.

It is to be noted that the parallelogram law, though only proved for motions, can be shown by similar methods to be applicable to the resolution and composition of velocities and accelerations.

13. ***Simple Harmonic Motion.***—If a point move in a

circle with a constant velocity, the point of intersection of a diameter and a perpendicular drawn from the moving point to this diameter will have a simple harmonic motion. Its velocity at any instant will be the velocity in the circle resolved at that instant parallel to the diameter. The radius of the circle is the amplitude of the motion. The period is the time between any two successive recurrences of a particular condition of the moving-point. The position of a point executing a simple harmonic motion can be expressed by the interval of time which has elapsed since the point last passed through the middle of its path in the positive direction. This interval of time, when expressed as a fraction of the period, is the *phase.*

We further define rotation in positive direction as that rotation in the circle which is contrary to the motion of the hands of a clock, or counter-clockwise. Motion from left to right in the diameter is also considered positive. Displacement to the right of the centre is positive, and to the left, negative.

If a point start from X (Fig. 8), the position of greatest positive elongation, with a simple harmonic motion, its distance from O or its *displacement* at the end of the time t, during which the point in the circle has moved through the arc BX, is $OC = OB \cos \phi$. Now, OB is equal to OX, the amplitude $= a$, and $\phi = \frac{2\pi t}{T}$, where T is the period; hence

$$s = a \cos \frac{2\pi t}{T}. \qquad (6)$$

To find the velocity at the point C, we must resolve the velocity of the point moving in the circle into its

components parallel to the axes. The component at the point C along OX is $V \sin\phi$; or since $V = \frac{2\pi a}{T}$,

$$= -\frac{2\pi a}{T}\sin\frac{2\pi t}{T}, \qquad (7)$$

remembering that motion from right to left is considered negative.

In order to find the acceleration at the point C directed towards O, we assume the equation for velocity

$$v = -\frac{2\pi a}{T}\sin\frac{2\pi t}{T}.$$

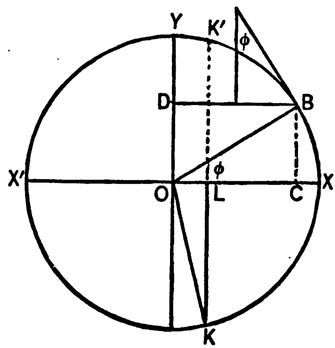

Fig. 8.

Since, if the point is moving with an acceleration, the velocity increases with the time, as the time increases by a small increment Δt, the velocity also increases by the increment Δv. The equation then becomes

$$v + \Delta v = -\frac{2\pi a}{T}\left[\sin\left(\frac{2\pi t}{T} + \frac{2\pi\Delta t}{T}\right)\right]$$
$$= -\frac{2\pi a}{T}\left(\sin\frac{2\pi t}{T}\cdot\cos\frac{2\pi\Delta t}{T} + \cos\frac{2\pi t}{T}\sin\frac{2\pi\Delta t}{T}\right).$$

When Δt becomes very small, $\cos\frac{2\pi\Delta t}{T}$ approaches the limit unity, and $\sin\frac{2\pi\Delta t}{T}$ can be replaced by its arc $\frac{2\pi\Delta t}{T}$; making these changes, and transposing,

$$\frac{\Delta v}{\Delta t} = -\frac{4\pi^2 a}{T^2}\cos\left(\frac{2\pi t}{T}\right);$$

but $\frac{\Delta v}{\Delta t}$ represents acceleration; hence the acceleration is

$$f = -\frac{4\pi^2}{T^2} a \cos\left(\frac{2\pi t}{T}\right). \qquad (8)$$

It is important to notice that the acceleration in a simple harmonic motion is proportional to the displacement. It is of the opposite sign from the displacement; that is, acceleration to the right of O is negative, and to the left of O positive.

It is often necessary to reckon time from some other position than that of greatest positive elongation. In that case the time required for the moving-point to reach its greatest positive elongation from that position, or the angle described by the corresponding point in the circumference in that time, is called the *epoch* of the new starting-point. In determining the epoch, it is necessary to consider, not only the position, but the direction of motion, of the moving-point at the instant from which time is reckoned. Thus, if L, corresponding to K in the circumference, be taken as the starting-point, the epoch is the time required to describe the path LX. But if L correspond to the point K' in the circumference, the motion in the diameter is negative, and the epoch is the time required for the moving-point to go from L through O to X' and back to X.

The epochs in the same cases, expressed in angle, are, first, the angle measured by the arc KX; and, second, the angle measured by the arc $K'X'KX$.

Choosing K in the circle, or L in the diameter, as the point from which time is to be reckoned, the angle ϕ equals angle KOB — angle KOX, or $\frac{2\pi t}{T} - \epsilon$, where

t is now the time required for the moving-point to describe the arc KB, and ϵ is the epoch or the angle KOX.

The formulas then become

$$s = a\cos\left(\frac{2\pi t}{T} - \epsilon\right)$$

$$v = -\frac{2\pi a}{T}\sin\left(\frac{2\pi t}{T} - \epsilon\right)$$

$$f = -\frac{4\pi^2}{T^2}a\cos\left(\frac{2\pi t}{T} - \epsilon\right).$$

Returning to our first suppositions, letting X be the point from which epoch and time are reckoned, it is plain, that, since

$$BC = a\sin\phi = a\cos\left(\phi - \frac{\pi}{2}\right) = a\cos\left(\frac{2\pi t}{T} - \frac{\pi}{2}\right),$$

the projection of B on the diameter OY also has a simple harmonic motion, differing in epoch from that in diameter OX by $\frac{1}{4}T$. It follows immediately, that the composition of two simple harmonic motions at right angles to one another, having the same amplitude and the same period, and differing in epoch by one quarter the period, will produce a motion in a circle of radius a with a constant velocity. More generally the co-ordinates of a point moving with two simple harmonic motions at right angles to one another are

$$x = a\cos(\phi - \epsilon),\ \text{and}\ y = b\cos\phi'.$$

If ϕ and ϕ' are commensurable, that is, if $\phi' = n\phi$, the curve is re-entrant. Making this supposition,

$$x = a\cos\phi\cos\epsilon + a\sin\phi\sin\epsilon,\ \text{and}\ y = b\cos n\phi.$$

Various values may be assigned to a and to b and to n. Let a and b be equal and $n = 1$; then

$$x = y \cos \epsilon + (a^2 - y^2)^{\frac{1}{2}} \sin \epsilon;$$

from which

$$x^2 - 2xy \cos \epsilon + y^2 \cos^2 \epsilon = a^2 \sin^2 \epsilon - y^2 \sin^2 \epsilon,$$

or,

$$x^2 - 2xy \cos \epsilon + y^2 = a^2 \sin^2 \epsilon.$$

This becomes, when $\epsilon = 90°$, $x^2 + y^2 = a^2$, the equation for a circle; and if $\epsilon = 0°$, the equation for a straight line through the origin, making an angle of 45° with the axis of X. With intermediate values of ϵ, it is the equation for an ellipse. If we make $n = 2$, we obtain, as special cases of the curve, a parabola and a lemniscate, according as $\epsilon = 0°$ or $90°$. If a and b are unequal, and $n = 1$, we get, in general, an ellipse.

If a line in which a point is describing a simple harmonic motion be made to move in a direction perpendicular to itself, the moving-point will describe an harmonic curve, called also a curve of sines. It is in outline a simple wave. Examples of such curves are found in the waves of the ether which constitute light. If the ordinates of the curve be taken, not to represent displacement transversely from a fixed line, but displacement longitudinally from points of equilibrium along a fixed line, the curve may be employed to represent the longitudinal vibrations of the air when transmitting sound. The length of the wave is the distance traversed by the moving line during the period of the simple harmonic motion, and is, therefore, the distance between any two identical conditions of points on the line of progress of the wave. The amplitude of the wave is the amplitude of the simple harmonic motion.

If we assume the origin of co-ordinates such that the epoch of the simple harmonic motion at the axis of ordinates is 0, the displacement from the line of progress of any point on the wave is represented by

$$s = a\cos\left(2\pi\frac{t}{T}\right).$$

The displacement due to any other wave differing from the first only in the epoch is represented by

$$s_1 = a\cos\left(2\pi\frac{t}{T} - \epsilon\right).$$

We shall now show, in the simplest case, the result of compounding two wave motions.

The displacement due to both waves is, of course,

$$s + s_1 = a\left[\cos 2\pi\frac{t}{T} + \cos\left(2\pi\frac{t}{T} - \epsilon\right)\right]$$

$$= a\left(\cos 2\pi\frac{t}{T} + \cos 2\pi\frac{t}{T}\cos\epsilon + \sin 2\pi\frac{t}{T}\sin\epsilon\right)$$

$$= a\left[\cos 2\pi\frac{t}{T}(1 + \cos\epsilon) + \sin 2\pi\frac{t}{T}\sin\epsilon\right].$$

If we assume a value A, such that

$$A\cos\phi = a(1 + \cos\epsilon),$$

and

$$A\sin\phi = a\sin\epsilon,$$

we may represent the last value of $s + s_1$ by

$$A\cos\left(2\pi\frac{t}{T} - \phi\right).$$

From the two equations containing A, we obtain, by adding the squares of the values of $A\sin\phi$ and $A\cos\phi$,

$$A = (2a^2 + 2a^2\cos\epsilon)^{\frac{1}{2}};$$

and, by dividing the value of $A \sin \phi$ by that of $A \cos \phi$, we obtain

$$\phi = \tan^{-1} \frac{\sin \epsilon}{1 + \cos \epsilon}.$$

The displacement thus becomes

$$S = a(2 + 2 \cos \epsilon)^{\frac{1}{2}} \cos\left(2\pi \frac{t}{T} - \tan^{-1} \frac{\sin \epsilon}{1 + \cos \epsilon}\right). \quad (9)$$

This equation is of great value in the discussion of problems in optics.

The principle suggested in the above discussion, that the resultant of the composition of two simple harmonic motions is an harmonic motion whose elements depend on those of the components, was proved by Fourier to hold generally.

Fourier's theorem may be stated as follows: "Any complex periodic function may be resolved into a number of simple harmonic functions whose periods are commensurable with that of the original function."

As an example, any wave not simple may be decomposed into a number of simple waves whose lengths are to each other as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc. The number of these simple waves is, in general, infinite, but in special cases determinate both as to number and to period.

14. ***Force.*** — Whenever any change occurs, or tends to occur, in the momentum of a body, we ascribe it to a cause called a **force.**

Whenever motions of matter are effected by our direct personal effort, we are conscious, through our muscular sense, of a resistance to our effort, involving an expenditure of muscular energy to overcome it. The conception of force to which this consciousness gives rise, we

transfer, by analogy, to the interaction of any bodies which is or may be accompanied by change of motion. The question whether this analogy is or is not valid, is not involved in a purely physical discussion of the subject. A force, in the physical sense, is the assumed cause of an observed change of motion. It is known and measured solely by that change. It is, therefore, if constant, fully expressed by the acceleration which it will impart to unit mass. If momentary or variable, it is measured, at any instant, by the change of momentum which it would cause in unit time if during that time it were to remain constant. The formula for force is, therefore,

$$m\frac{v - v_0}{t} = mf. \qquad (10)$$

Its dimensions are MLT^{-2}.

As acceleration is always referred to some fixed direction, it follows that force is a quantity having direction.

The product of the time during which a force acts by its mean intensity is called the impulse of the force.

The practical unit of the force is the dyne, which is the force that can impart to a gram of matter one unit of acceleration; that is to say, one unit of velocity in one second.

15. ***Field of Force.*** — A field of force is a region such that a particle constituting a part of a mutually interacting system, placed at any point in the region, will be acted on by a force, and will move, if free to do so, in the direction of the force. The particle so moving would, if it had no inertia, describe what is called a line of force, the tangent to which, at any point, is the direction of the force at that point. The strength of

the field at a point is measured by the force developed by unit quantity at that point, and is expressible, in terms of lines of force, by the convention that each line represents a unit of force, and that the force acting on unit quantity at any point varies as the number of lines of force which pass perpendicularly through unit area at that point. Each line, therefore, represents the direction of the force, and the number of lines in unit area the strength of field. An assemblage of such lines of force considered with reference to their bounding-surface is called a tube of force.

16. ***Newton's Laws of Motion.*** — We are now ready for the consideration of the laws of motion, first formally enunciated and successfully applied by Newton, and hence known by his name: —

LEX I. — Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus illud a viribus impressis cogitur statum suum mutare.

LEX II. — Mutationem motus proportionalem esse vi motrici impressae & fieri secundum lineam rectam qua vis illa imprimitur.

LEX. III. — Actioni contrariam semper & aequalem esse reactionem; sive corporum duorum actiones in se mutuo semper esse aequales & in partes contrarias dirigi.

The subjoined translations are given by Thomson and Tait: —

LAW I. — Every body continues in its state of rest or of motion in a straight line, except in so far as it may be compelled by force to change that state.

LAW II. — Change of motion is proportional to force

applied, and takes place in the direction of the straight line in which the force acts.

LAW III. — To every action there is always an equal and contrary reaction: or, the mutual actions of any two bodies are always equal and oppositely directed.

17. ***Discussion of the Laws of Motion.*** — (1) The first law is a statement of the important truths implied in our definition of force, — that motion, as well as rest, is a natural state of matter; that moving bodies, when entirely free to move, proceed in straight lines, and describe equal spaces in equal times; and that force is the cause of any deviation from this uniform rectilinear motion.

That a body at rest should continue indefinitely in that state seems perfectly obvious as soon as the proposition is entertained; but that a body in motion should continue to move in a straight line is not so obvious, since motions with which we are familiar are frequently arrested or altered by causes not at once apparent. This important truth, which is forced upon us by observation and experience, may, however, be presented so as to appear almost self-evident; for if we conceive of a body moving in empty space, we can think of no reason why it should alter its path or its rate of motion in any way whatever.

(2) The second law presents, first, the proposition on which the measurement of force depends; and, secondly, states the identity of the direction of the change of motion with the direction of the force. Motion is here synonymous with momentum as before defined. The first proposition we have already employed in deriving the formula representing force. The second,

with the further statement that more than one force can act on a body at the same time, leads directly to a most important deduction respecting the combination of forces; for the parallelogram law for the resolution and composition of motions being proved, and forces being proportional to and in the same direction as the motions which they cause, it follows, if any number of forces acting simultaneously on a body be represented in direction and amount by lines, that their resultant can be found by the same parallelogram construction as that which serves to find the resultant motion; whence we derive the parallelogram and the polygon of forces.

In case the resultant of the forces acting on a body be zero, the body is said to be in equilibrium.

(3) When two bodies interact so as to produce, or tend to produce, motion, their mutual action is called a stress. If one body be conceived as acting, and the other as being acted on, the stress, regarded as tending to produce motion in the body acted on, is a force. The third law states that all interaction of bodies is of the nature of stress, and that the two forces into which the stress can be resolved are equal and oppositely directed.

From this follows directly the deduction, that the total momentum of a system is unchanged by the interaction of its parts; that is, the momentum gained by one part is counterbalanced by the momentum lost by the others. This principle is known as the conservation of momentum.

18. ***Collision of Bodies.***—If two bodies, m_1 and m_2, with velocities v_1 and v_2 in the same line, impinge, their velocities after contact are found, in two extreme cases, as follows:—

(1) If the bodies are perfectly inelastic, there is no tendency for them to separate, their final velocities will be equal, and their momentum will be equal to the sum of their separate momenta; hence

$$m_1v_1 + m_2v_2 = (m_1 + m_2)x, \qquad (11)$$

where x is the velocity after impact.

(2) If the bodies are perfectly elastic, they tend to separate with a force equal to that by which they are compressed.

Let v represent their common velocity just at the instant when the resistance to compression balances the impulsive force. Then the change in momentum in each body up to this instant is $m_1(v_1 - v)$, or $m_2(v - v_2)$; and the further change of momentum, by reason of the elasticity of the bodies, is the same, whence the whole momentum lost by the one is $2m_1(v_1 - v)$ and that gained by the other $2m_2(v - v_2)$; that is,

$$m_1v_1 - m_1x = 2m_1v_1 - 2m_1v,$$

if x represent the final velocity of m_1. From this

$$x = 2v - v_1.$$

In like manner, y representing the final velocity of m_2, we find

$$y = 2v - v_2.$$

From the formula for inelastic bodies, which is applicable at the moment when both bodies are moving with the same velocity,

$$v = \frac{m_1v_1 + m_2v_2}{m_1 + m_2};$$

whence, finally,

$$x = \frac{(m_1 - m_2)v_1 + 2m_2v_2}{m_1 + m_2},$$

$$y = \frac{(m_2 - m_1)v_2 + 2m_1v_1}{m_1 + m_2}. \qquad (12)$$

19. ***Inertia.***—The principle of equality of action and re-action holds equally well when we consider a single body as acted on by a force. The resistance to change of motion offered by the inertia of the body is equal in amount and opposite in direction to the acting force. Inertia is not of itself a force, but the property of a body, enabling it to offer a resistance to a change of motion.

20. ***Work and Energy.***—When a force causes motion through a space, it is said to do work.

The measure of work is the product of the force and the space traversed by the body on which the force acts. The formula expressing work is evidently

$$w = Fs = mfs. \qquad (13)$$

Its dimensions are ML^2T^{-2}.

It appears, then, that, in the defined sense of the term, no work is done upon a body by a force unless it is accompanied by a change of position; and, further, that the amount of work is independent of the time taken to perform it. Both of these statements need to be made, because of a natural tendency to confound work with conscious effort, and to estimate it by the effect on our system.

A body may, in consequence of its motion or position with respect to other bodies, have a certain capacity

for doing work. This capacity for doing work is its energy. Energy is of two kinds, usually distinguished as potential and kinetic. The former is due to the position of the body, the latter to its motion.

If a mass m be moving with a velocity v, its capacity for doing work may be determined from the consideration, that, if the motion be opposed by a force F equal to mf, the mass suffers a negative acceleration, and is finally brought to rest after traversing a space s equal to $\frac{v^2}{2f}$, in opposition to the force. (See Eq. 5, p. 19).

Multiplying both sides of this equation by $F = mf$, we have $Fs = \frac{mv^2}{2}$. But Fs is the work done by the body against the force F, and is, therefore, the capacity which the body originally had for doing work. This capacity—that is, the kinetic energy of the body—is then represented by the expression $\frac{mv^2}{2}$.

It can also be shown that the difference between the kinetic energy of a body at the beginning and at the end of a given space is equal to the work done in traversing that space. For, if we consider the mass m having an acceleration f, and moving through a space s, so small that the acceleration may be assumed constant, we have, from Eq. 5, p. 18,

$$v^2 = v_0^2 \pm 2fs,$$

whence

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mv_0^2 \pm mfs,$$

the plus or minus sign being used according as the work is done by or against the force. Since any motion whatever may be divided into portions in which the

above conditions hold true, it follows that we have finally, for any motion,

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mv_0^2 \pm mf_1s_1 \pm mf_2s_2 \pm \ . \ . \ . = \tfrac{1}{2}mv_0^2 \pm \Sigma mfs.$$

Since $\frac{1}{2}mv_0^2$, or the initial kinetic energy, is a constant quantity, it follows that $\frac{1}{2}mv^2 \mp \Sigma mfs$, or the sum of the kinetic and potential energies, is a constant quantity in any system where the total energy exists in these two forms. In other words, a body, by losing potential energy, gains an equal amount of kinetic energy; and the kinetic energy, being used to do work against acceleration, places the body in a position where it again possesses its original amount of potential energy.

So, evidently, if a body start from rest, the kinetic energy acquired by motion through a certain distance is equal to the potential energy of the body at rest due to the possibility of its traversing that distance.

Since the dimensions of energy are expressed in every case by ML^2T^{-2}, and since the square of a length cannot involve direction, it follows that energy is a quantity independent of direction.

The practical unit of energy is the *erg*, and is the energy of a body so conditioned that it can exert a force of one dyne through a space of one centimetre.

21. ***Difference of Potential.*** — In the discussion which follows, we deal with forces which vary directly as the product of the quantities acting, and inversely as the squares of the distances which separate them. The difference of potential between two points in a field of force is measured by the work done in moving a test unit of the quantity to whose presence the force is due, from one point to the other.

If the field is due to the presence of an acting quantity m, which repels the test unit, at a distance s, with a force expressed by

$$\frac{Km}{s^2},$$

the difference of potential between two points distant, r and R, from the acting quantity m is expressed by

$$V_r - V_R = Km\left(\frac{1}{r} - \frac{1}{R}\right).$$

The symbol K represents the force with which two unit masses at unit distance repel one another.

To obtain this formula in the simplest case, let us suppose a body at the point O (Fig. 9) acting upon a unit at P

Fig. 9.

with a force equal to $\frac{Km}{OP^2}$. If the body move to Q, the force at Q is $\frac{Km}{OQ^2}$; and the average force acting while the body is moving in the path PQ, provided this path be taken small enough, is

$$\frac{Km}{OP \cdot OQ}.$$

The work done through PQ is

$$\left(\frac{Km}{OP \cdot OQ}\right)PQ = \frac{Km}{OP \cdot OQ}(OP - OQ) = Km\left(\frac{1}{OQ} - \frac{1}{OP}\right).$$

The work done in moving through any other small space QR towards S is, similarly,

$$Km\left(\frac{1}{OR} - \frac{1}{OQ}\right).$$

The last value obtained by moving from S to T is

$$Km\left(\frac{1}{OT} - \frac{1}{OS}\right).$$

The sum of these values,

$$Km\left(\frac{1}{OT} - \frac{1}{OP}\right),$$

gives the work done in passing over the space PT.

Designating OT as r and OP as R, and performing the operation indicated by the formula for every other acting quantity at O, we obtain the expression,

$$\Sigma Km\left(\frac{1}{r} - \frac{1}{R}\right)$$

for the difference of potential between P and T, where Σ is a summation sign.

Since the point T can be considered as on the surface of a sphere of which O is the centre, and since the force $\frac{m}{OT^2}$ acts along TO perpendicular to that surface, and cannot, therefore, have a component tending to produce motion on that surface, there is no work done in moving the unit at T over the surface of the sphere to any other point X on it: it follows that the difference of potential between P and any other point at distance r from O is the same as that between P and T.

$$V_r - V_R = Km\left(\frac{1}{r} - \frac{1}{R}\right). \qquad (14)$$

Such a surface as the one described, to which the lines of force are perpendicular, is called an equipotential surface.

It is further evident, that the amount of work done upon the unit to move it from P to T is independent of the path. For, if this were not so, it would be possible, by moving from P to T on one path, and returning on another, to accumulate an indefinite amount of energy; which the doctrine of the conservation of energy shows to be impossible.

If the point P be supposed to be at a distance from O so great that the force at that distance vanishes, it is then at zero potential. R becomes indefinitely large, and the absolute potential at T becomes

$$V_r = \Sigma\frac{Km}{r}. \tag{15}$$

This formula expresses the work necessary to move the unit against the repulsion of the system at O up to the point T from an infinite distance. If the acting quantity attract the unit, the work is done by the attraction upon the body in so moving up to T, and the potential is negative.

From the definitions, it is plain that the difference of potential between P and T equals the difference of the potential energies of a unit at those points.

The amount of work done in moving a unit from P to Q, and hence the difference of potential between P and Q, is represented by

$$V_P - V_Q = \frac{Km}{OP \cdot OQ} PQ,$$

where $\frac{Km}{OP \cdot OQ}$ represents the average force. From this relation we have

$$\frac{Km}{OP \cdot OQ} = \frac{V_P - V_Q}{PQ},$$

or, if PQ be taken small enough,

$$\frac{Km}{OP^2} = \frac{V_P - V_Q}{PQ},$$

whence the force in any one direction at a point in the field of force is equal to the rate of change of potential in that direction, at that point, with respect to space.

22. ***Theorems relating to Difference of Potential.*** — (1) The force at any point within a spherical shell of uniform thickness and density of acting material is zero. For, at the point O (Fig. 10) within the sphere, the force, if an attraction, acting towards a is $\frac{\text{mass } ab_1}{Oa^2}$, and towards c is $\frac{\text{mass } cd_1}{Oc^2}$; so that the efficient force tending to produce motion, say, towards a, is expressed by

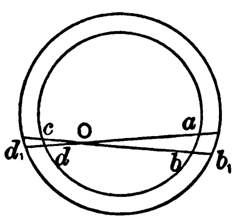

Fig. 10.

$$\frac{\text{mass } ab_1}{Oa^2} - \frac{\text{mass } cd_1}{Oc^2}.$$

Now, if ab_1, cd_1, be taken small enough, they will be frusta of similar cones, and, as a consequence,

$$\frac{ab_1}{Oa^2} = \frac{cd_1}{Oc^2};$$

from which, since the density of the shell is uniform,

$$\frac{\text{mass } ab_1}{Oa^2} - \frac{\text{mass } cd_1}{Oc^2} = 0.$$

Since the whole surface of the shell may be cut by similar cones, for which similar equations will hold, the total force exerted by the shell on a particle within it becomes zero. This being so, it follows that the potential throughout the sphere is constant; for no work is

required to move the unit from one point to another in the interior.

(2) The potential, and therefore the force at a point due to the presence of a uniform spherical shell of acting material, depends only on the quantity of material and on the distance of the point considered, from the centre of the sphere. In the figure (Fig. 11), let CKL

Fig. 11.

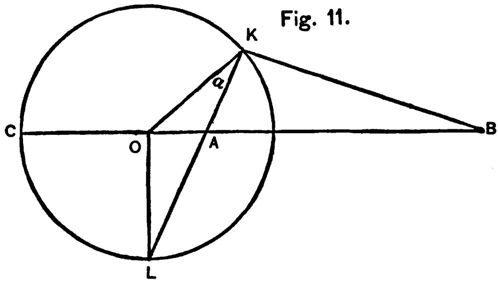

Fig. 11.

represent a central section of the shell, of which O is the centre. Let d represent the mass of unit of area. The potential at B, due to the element s of the sphere at K, is $V_{\prime} = \frac{ds}{BK}$; and the potential due to the whole sphere is the summation of that due to all the similar elements making up the sphere. Take a point A on the line OB, such that $OA \cdot OB = R^2$, where R is the radius of the sphere, and draw AK, produce it to L, and draw OK and OL. Now, if we represent the angle OKA by α, and the solid angle subtended by the element s, as seen from A, by ω, we may express s in other terms as $\frac{AK^2 \cdot \omega}{\cos \alpha}$; hence

$$V_{\prime} = \frac{dAK^2\omega}{BK \cos \alpha}.$$

Now, since, by construction, $OB : R = R : OA$, and the angle KOB is common to the two triangles KOA and BOK, these triangles are similar; hence

$$BK : AK = OB : OK,$$

whence

$$\frac{AK}{BK} = \frac{R}{OB}.$$

The value of the potential due to K may be written

$$V_{,} = d\frac{AK}{\cos a} \cdot \frac{R}{OB} \cdot \omega.$$

The value for the potential of the element at L is, similarly,

$$V_{,,} = d\frac{AL}{\cos a} \cdot \frac{R}{OB} \cdot \omega.$$

Adding these values, we obtain

$$V_{,} + V_{,,} = d\frac{R}{OB} \cdot \omega\left(\frac{AK + AL}{\cos a}\right).$$

But

$$\frac{AK + AL}{\cos a} = \frac{KL}{\cos a} = 2\,OK = 2R;$$

hence we obtain, finally,

$$V_{,} + V_{,,} = 2d\frac{R^2 \cdot \omega}{OB}.$$

Now the sphere can be divided into two portions, made up of elements similar to K and L, by a plane passing through A normal to OB. We obtain the whole potential, therefore, by summing all the potential values due to these pairs of elements, whence

$$V = 2d\frac{R^2}{OB}\Sigma\omega.$$

The sum of all the elementary solid angles on one side of the plane from the point A in it is, of course, 2π; hence, finally,

$$V = 4\pi \frac{R^2 d}{OB} = \frac{M}{OB},$$

where M is the mass of the spherical shell.

Since the force at the point B depends on the rate of change of potential at that point, it varies inversely as the square of the distance OB. Represent OB by l. In the expression $V = \frac{m}{l}$, let l change by a small increment Δl, and denote the corresponding change in the potential by ΔV; then

$$V + \Delta V = \frac{m}{l + \Delta l},$$

$$Vl + V\Delta l + \Delta V l + \Delta V \cdot \Delta l = m.$$

As ΔV and Δl are both very small quantities, their product may be neglected. We then have

$$V \cdot \Delta l + \Delta V \cdot l = 0,$$

or

$$\frac{\Delta V}{\Delta l} = -\frac{V}{l} = -\frac{m}{l^2}.$$

The force, therefore, at a point outside a sphere of uniform density varies inversely as the square of its distance from the centre of the sphere. This result enables us to deal with spheres of gravitating matter, or spherical shells, upon which is a uniform distribution of electricity, as if they were gravitating or electrified points.

(3) If, in the last proposition, we let $l = R$, we obtain for the value of the force just outside the shell

$$-\frac{4\pi R^2 d}{R^2} = 4\pi d.$$

Since the force just inside the shell vanishes, in consequence, as we have seen, of the equal and opposite actions of the portions of the sphere ab and acb (Fig. 12), and since the total force at the point P outside the sphere is $4\pi d$, it follows that the force at P, due to ab, is $2\pi d$. If the radius be taken large enough, ab may be considered as flat, and constituting a disk: hence the force at a point near a flat disk of density d is $2\pi d$. Since the force at a point near one surface of the disk is $2\pi d$ in one direction, and near the other surface $2\pi d$ in the other direction, it is clear, that, in passing through the disk, the force changes by $4\pi d$.

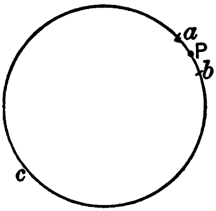

Fig. 12.

23. ***Moment of Force.*** — The moment of force about a point is defined as the product of the force and the perpendicular drawn from the point upon the line of direction of the force.

The moment of a force, with respect to a point, measures the value of the force in producing rotation about that point.

If momentum be substituted for force in the foregoing definition, we obtain the definition of moment of momentum.

In order to show that the moment of a force measures the value of that force in producing rotation, we will find the direction and amount of the resultant of two forces acting on a rigid bar, but not applied at the same point.

Let BD (Fig. 13) be the bar, DF and BG the forces. Their lines of direction will, in general, meet at some

point, as O. Moving the forces up to O, and applying the parallelogram law, we obtain the resultant OJ, which cuts the bar at A. If we resolve both forces separately, parallel to OJ and BD, this resultant equals in amount the sum of those components taken parallel to OJ. Hence the components EF and CG, taken parallel to DB, annul one another's action, and, being in opposite directions, are equal. Now, by similarity of triangles,

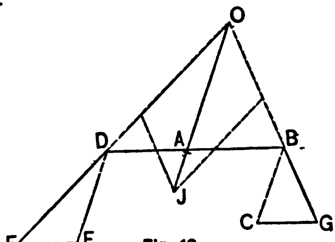

Fig. 13.

$$OA : AB = BC : CG,$$

and

$$OA : AD = DE : EF;$$

whence

$$AB \cdot BC = AD \cdot DE;$$

and resolving both DE and BC perpendicular to DB, we see that the moments of force about A are equal. Now, if the resultant OJ be antagonized by an equal and opposite force applied at A, there will be no motion. Hence the tendencies to rotation due to the forces are equal, — a result which is in accord with our statement that the moment of force is a measure of the value of the force in producing rotation.

24. ***Couple.*** — The combination of two forces, equal and oppositely directed, acting on the ends of a rigid bar, is called a couple. By the preceding proposition, the resultant of these forces vanishes, and the action of a couple does not give rise to any motion of translation. The forces, however, conspire to produce rota-

tion about the mid-point of the bar. It follows from the fact that a couple has no resultant, that it cannot be balanced by any single force.

25. ***Moment of Couple.*** — The moment of a couple is the product of either of the two forces into the perpendicular distance between them. It follows from what has been already proved, that this measures the value of the couple as respects rotation.

26. ***Centre of Inertia.*** — If we consider any system of equal material particles, the point whose distance from any plane is equal to the average distance of the several particles from that plane is called the centre of inertia. This point can be shown to be perfectly definite for any system of particles. It follows from the definition, that, if the plane pass through the centre of inertia, the sum of the distances of the particles on one side of the plane, from the plane, will be equal to the sum of the distances of the particles on the other side: hence, if the particles are all moving with a common velocity parallel to the plane, the sum of the moments of momentum on the one side is equal to the sum of the moments of momentum on the other side. And, further, if the particles all have a common acceleration, or are each acted on by equal and similarly directed forces, the sum of the moments of force on one side is equal to the sum of the moments of force on the other side.

If we combine the forces acting on two of the particles, one on each side of the plane, we obtain a resultant equal to their sum, whose distance from the plane is determined by the distances of the two particles from the plane. Combining this resultant with the force on

another particle, we obtain a second resultant; and, by continuing this process until all the forces have been combined, we obtain a final resultant, equal to the sum of all the forces lying in the plane, and passing through the centre of inertia. This resultant expresses, in amount, direction, and point of application, the force which, acting on a mass equal to the sum of all the particles situated at the centre of inertia, would impart the same acceleration to it as the conjoined action of all the separate forces on the separate particles imparts to the system.

When the forces do not act in parallel lines, the proposition just stated does not hold true, except in special cases. Bodies in which it still holds are for that reason called *centrobaric* bodies.

The centre of inertia can be readily found in most of the simple geometrical figures. For the sphere, ellipsoid of revolution, or parallelopiped, it evidently coincides with the centre of figure; for a plane passing through that point in each case cuts the solid symmetrically.

27. **Mechanical Powers.** — The preceding definitions and propositions find their most elementary application in the so-called mechanical powers.

These are all designed to enable us, by the application of a certain force at one point, to obtain at another point a force, in general not equal to the one applied. Six mechanical powers are usually enumerated, — the lever, pulley, wheel and axle, inclined plane, wedge, and screw.

(1) *The Lever* is any rigid bar, whose weight may be neglected, resting on a fixed point called a fulcrum.

From the proposition under "Moments of Force," it may be seen, that, if forces be applied to the ends of the lever, there will be equilibrium when the resultant passes through the fulcrum. In that case the moments of force about the fulcrum are equal; whence, if the forces act in parallel lines, it follows that the force at one end is to the force at the other end in the inverse ratio of the lengths of their respective lever-arms. If l and l_1 represent the lengths of the arms of the lever, and P and P_1 the forces applied to their respective extremities, then $Pl = P_1 l_1$.

The principle of the equality of action and re-action enables us to substitute for the fulcrum a force equal to the resultant of the two forces. We have then a combination of forces as represented in the diagram (Fig. 14). Plainly any one of these forces may be considered as taking the place of the fulcrum, and either of the others the power or the weight.

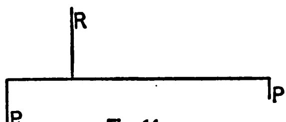

Fig. 14.

The lever is said to be of the first kind if R is fulcrum and P power, of the second kind if P_1 is fulcrum and P power, of the third kind if P is fulcrum and R power.

(2) *The Pulley* is a frictionless wheel, in the groove of which runs a perfectly flexible, inextensible cord.

If the wheel is on a fixed axis, the pulley merely changes the direction of the force applied at one end of the cord. If the wheel is movable and one end of the cord fixed, and a force be applied to the other end parallel to the direction of the first part of the cord,

the force acting on the pulley is double the force applied: for the stress on the cord gives rise to a force in each branch of it equal to the applied force; each of these acts on the wheel, and, since the radii of the wheel are equal, the resultant of these two forces is a force equal to their sum applied at the centre of the wheel. From these facts the relation of the applied force to the force obtained in any combination of pulleys is evident.

(3) *The Inclined Plane* is any frictionless surface, making an angle with the line of direction of the force applied at a point upon it. Resolving the force P (Fig. 15), making an angle ϕ, with the normal to the plane, into its components $P\cos\phi$ and $P\sin\phi$ perpendicular to and parallel with the plane, $P\sin\phi$ is alone effective to produce motion. Consequently, a force $P\sin\phi$ acting parallel to the surface will balance a force P, making an angle ϕ, with the normal to the surface. If the plane be taken as the hypothenuse of a right-angled triangle ABC, of which the base AB is perpendicular to the line of direction of the force, then, by similarity of triangles, angle $BAC = \phi$: whence the force obtained parallel to AC is equal to the force applied multiplied by the sine of the angle of inclination of the plane. If the components of the force applied be taken, — the one, as before, perpendicular to the plane AC, and the other parallel to the base AB, — the force obtained parallel to AB is equal to the force applied multiplied by the tangent of the angle of inclination of the plane.

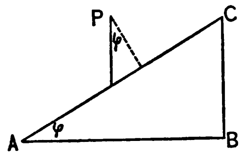

Fig. 15.

(4) *The Wheel and Axle* is essentially a combination of levers.

(5) *The Wedge* is made up of two similar inclined planes set together, base to base.

(6) *The Screw* is a combination of the lever and the inclined plane.

The special formulas expressing the relations of the force applied to the force obtained by the use of these combinations, are deduced from those for the more elementary mechanical powers.

It may be seen, in general, in the use of the mechanical powers, that the force applied is not equal to the force obtained. A little consideration will show, however, that the energy expended is always equal to the work done.

Any arrangement of the mechanical powers designed to transmit work, undiminished, is called a machine. The more nearly this design is realized in actual combinations of materials, the more closely the machine approaches perfection. The elasticity of the materials we are compelled to employ, friction, and other causes which modify the conditions required by theory, make the attainment of such perfection impossible.

The ratio of the useful work done to the energy expended is called the efficiency of the machine. Since in every actual machine there is a loss of energy in the transmission, the efficiency is always a proper fraction.

28. ***Angular Velocity.*** — The angle contained by the line passing through two points, one of which is in motion, and any assumed line passing through the fixed point, will, in general, vary. The rate of its change is called the angular velocity of the moving-point. If ϕ

denote the angle swept out in the time t, then the angular velocity

$$\omega = \frac{\phi}{t}. \qquad (16)$$

The angular acceleration is the rate of change of angular velocity, and, if constant, is measured by

$$\psi = \frac{\omega - \omega_0}{t}. \qquad (17)$$

If the radian, or that arc of a circle which is equal to the length of its radius, be taken as the unit of angle, the dimensions of angle become

$$\frac{\text{arc}}{\text{radius}} = \frac{L}{L} = 1.$$

Hence the dimensions of angular velocity are $\frac{1}{T}$, and of angular acceleration, $\frac{1}{T^2}$.

If any point be revolving about a fixed point as a centre, its velocity in the circle varies as its angular velocity and the length of the radius jointly.

Angular velocities may be compounded by a process similar to that employed for the compositions of motions.

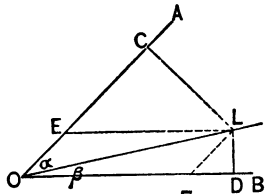

Fig. 16.

Let OA and OB (Fig. 16) represent two axes of rotation, about which points are revolving with angular velocities, ω_1 and ω_2 respectively; both rotations being clockwise when seen from the point O. The velocity at a point L, at unit distance from O, due to the motion about OA, is $\omega_1 \sin \alpha$, and that

due to motion about OB is $\omega_2 \sin\beta$ in the opposite direction. The whole velocity of L is, therefore,

$$\omega_1 \sin\alpha - \omega_2 \sin\beta.$$

There must be some position of L for which this velocity becomes zero. Then $\omega_1 \sin\alpha = \omega_2 \cdot \sin\beta$. It follows at once that every point on the line OL is at rest. If we consider OL as the axis of rotation, and suppose the angular velocity of every point of the system about this axis to be ω, such that $\omega \sin\alpha = \omega_2 \sin(\alpha + \beta)$, this angular velocity will give the actual velocity of any point. For example,

$$\omega \sin\beta = \omega_2 \frac{\sin(\alpha + \beta)}{\sin\alpha} \sin\beta$$

is the velocity at B; for the velocity at B is only due to rotation about OA, and is therefore given by $\omega_1 \sin(\alpha + \beta)$. From our previous equation,

$$\omega_1 \sin\alpha = \omega_2 \sin\beta;$$

hence

$$\omega_1 \sin(\alpha + \beta) = \omega_2 \frac{\sin(\alpha + \beta)}{\sin\alpha} \sin\beta.$$

Now

$$\omega \sin\beta = \omega_2 \frac{\sin(\alpha + \beta)}{\sin\alpha} \sin\beta,$$

and the equality of the expressions is shown. In a similar manner it may be shown that the value of the angular velocity at any point N as given by the expression $\omega \sin NOL$ is equal to that given by $\omega_1 \sin AON - \omega_2 \sin BON$. The two rotations about OA and OB may thus be combined to form one rotation about OL.

Draw LF and LE parallel to OA and OB. The lines

OE and OF are numerically proportional to the angular velocities ω_1 and ω_2: for, since $OELF$ is a parallelogram,

$$OE \sin \alpha = OF \sin \beta,$$

or,

$$\frac{\sin \alpha}{\sin \beta} = \frac{OF}{OE}.$$

But, from our first equation,

$$\frac{\sin \alpha}{\sin \beta} = \frac{\omega_2}{\omega_1};$$

whence

$$OF : OE = \omega_2 : \omega_1.$$

The line OL is likewise proportional to ω, for, from the figure,

$$OF : OL = \sin \alpha : \sin(\alpha + \beta);$$

whence we see immediately from the equation giving the value of ω, that

$$OL : OF = \omega : \omega_2.$$

We can therefore obtain the direction of the resultant axis, and the amount of the angular velocity due to rotation about two other axes, by laying off on those axes, from their point of intersection, lengths numerically equal to the angular velocities about them, and drawing the resultant, — the diagonal of the parallelogram of which they are the sides. And so also any angular velocity may be resolved into three whose axes are at right angles to one another by employing the trigonometrical functions of the angles which its axis makes with the three component axes.

If a body be rotating about an axis, it is an immediate

consequence of Newton's first law that it will continue to rotate in the same plane, or in parallel planes, unless force is applied to it from without. In other words, the axis of rotation always remains parallel to itself. This property is of great importance in the discussion of some very interesting applications of the preceding principles, which we shall next consider.

(1) The first of these is the method employed by Foucault to determine by experiment the fact of the earth's rotation. His apparatus consisted of a spherical pendulum bob, suspended by a truly cylindrical wire, so that it could swing freely in any plane. It can easily be seen, that, if such a pendulum were set up at the pole and swung, it would preserve its plane of oscillation invariable, and the earth would turn around under it, so that in twenty-four hours the pendulum would seem to have traversed a complete circle in the direction of the sun's apparent motion. At any other point on the earth's surface the change in apparent direction of the plane of oscillation would not be so great. If ω represent the angular velocity of the earth, t the duration of the experiment, and ϕ the latitude, the angle α between the directions of the first and last swing, as marked on the earth's surface, is expressed by

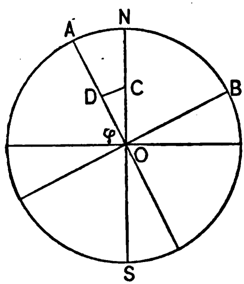

Fig. 17.

$$\alpha = \omega t \sin \phi.$$

The truth of this formula may be seen from the following demonstration: In the figure (Fig. 17), let the pendulum be supposed to be at A. Let NS be the earth's axis.

Now, the angular velocity ω, represented by OC, may be resolved into two components, OD in the direction of the force acting on the pendulum, and DC at right angles to it. The angular velocity DC has no influence in changing the relations of the pendulum and the earth; but the angular velocity $OD = OC \sin \phi = \omega \sin \phi$ is made evident by the rotation of a fixed line on the earth's surface, cutting the invariable plane of oscillation at the point of equilibrium of the pendulum. The plane of oscillation of the pendulum consequently appears to rotate in the opposite direction with an angular velocity $\omega \sin \phi$, and the angle swept out in any time t is $\omega t \sin \phi$. By such an apparatus has been determined, not only the fact of the earth's rotation, but even an approximate value of the length of the day.

(2) The phenomena presented by the gyroscope also offer an example of the application of the foregoing principles.

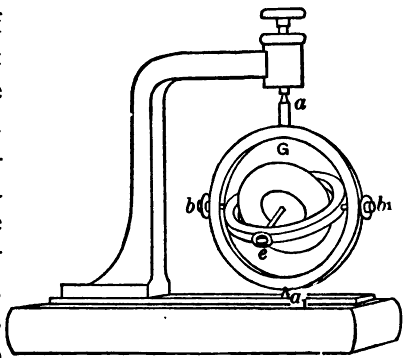

Fig. 18.

The construction of the apparatus can best be understood by the help of the diagram (Fig. 18). The outermost ring rests in a frame, and turns on the points a, a_1. The inner rests in the outer one, and turns on the pivots b, b_1, at right angles to the line of aa_1. Within this ring is mounted the wheel G, the axle of which is at right angles to the line bb_1, and therefore in the plane

of aa_1. At the point e is fixed a hook, from which weights may be hung. It is evident at once, that, if the wheel is mounted on the middle of the axle, the equilibrium of the apparatus is neutral in any position, and that a weight hung on the hook e would bring the axle of the wheel vertical, without moving the outer ring. If, however, the wheel be set in rapid rotation, with its axle horizontal, and a weight be hung on the hook, the whole system will revolve with a constant angular velocity about the points a, a_1, and the axle of the wheel will remain horizontal.

The explanation of this phenomenon follows from the principles which we have already discussed. The conditions given are, that a body rotating with an angular velocity in one plane is acted on by a force tending to produce rotation in a perpendicular plane.

Let the plane of the paper represent the horizontal plane, and the line AB (Fig. 19) represent the direction of the axle at any moment. Lay off on OA a length OP proportional to the angular velocity of the wheel. If B be the point of application of the weight, gravity tends to turn the system about an axis CD at right angles to AB. Let us suppose, first, that the system has an angular velocity about CD proportional to OQ. Compounding the two angular velocities OP and OQ, we obtain the resultant OR. Now, resolving OQ parallel and at right angles to OR, we see that the parallel component has been efficient in determining the length of

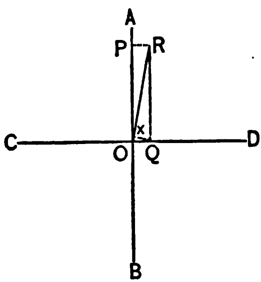

Fig. 19.

OR, the component at right angles to the direction of OR. In the limit, if OQ be made indefinitely small, the resolved component Ox parallel to OR vanishes in comparison with OQ; because from the triangles we have $\frac{OQ}{OR} = \frac{Ox}{OQ}$. The effect will be a change of direction of the axle AB in the horizontal plane, without a change in the angular velocity of the wheel. This change is the equivalent of the introduction of a new angular velocity about an axis perpendicular to the plane of the paper. This new angular velocity, compounded with the angular velocity about OA, gives rise, as before, to a change in the direction of the axis without a change in the angular velocity of the wheel; and this change in direction is such as to oppose the angular velocity about CD, introduced by the weight at B. The system will revolve in a horizontal plane about O as a centre.

Another explanation, leading to the same results, has been given by Poggendorff. As we have already shown, it requires the application of a force to change the direction of the axis of a rotating body. This force is expended in changing the direction of motion of the component parts of the body.

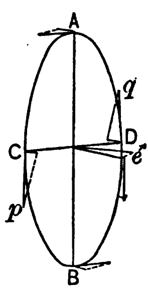

Fig. 20.

Poggendorff's explanation of the movements of the gyroscope is based on the action of couples formed by these separate forces. Let Fig. 20 represent the rotating wheel of the former diagram, the axle being supposed to be nearly horizontal. If the weight be hung at the point e, gravity tends to turn the wheel about a horizontal axis CD. The parti-

cles moving at A and at B in the plane CD offer no resistance to this change. Those at C moving downwards, and those at D moving upwards, act otherwise. The forces expressed by their momentum in the directions Cp and Dq are resolved into two each, — one of them in the new plane assumed by the wheel, and the other at right angles to it. It will be seen that the latter component acts at C towards the right, and at D towards the left. There is thus set up a couple acting to turn the system about the axis AB counter-clockwise, as seen from A. As soon as this rotation begins, the particles moving at A out of the paper, and at B through the paper, are turned out of their original directions, and there arises another couple, of which the component at A is directed towards the left, and at B towards the right. This couple tends to cause the system to rotate about the axis CD counter-clockwise, as seen from C, and thus to oppose the tendency to rotation due to the weight at e.

All other points on the wheel, except those in the lines AB and CD, are turned out of their paths by both rotations; and therefore components of the forces due to their motions appear in both couples in the final summation of effects. The result of the existence of these couples is a movement such as has already been described.

29. ***Moment of Inertia.***—The moment of inertia of any body about an axis is defined as the summation of the products of the masses of the particles making up the body and the squares of their respective distances from the axis.

This product is the measure of the importance of the body's inertia with respect to rotation, and is propor-

tional to the kinetic energy of the body having a given angular velocity about the axis; for if any particle m, at a distance r from the axis, rotate with an angular velocity ω, its velocity is $r\omega$ and its kinetic energy is $\frac{1}{2}m\omega^2r^2$. The whole kinetic energy of the body is, therefore, $\frac{1}{2}\omega^2\Sigma mr^2$; and since we have assumed $\frac{1}{2}\omega^2$ to be constant, Σmr^2 is proportional to the kinetic energy of the rotating body. If we can find a distance k such that $\frac{1}{2}k^2\omega^2\Sigma m = \frac{1}{2}\omega^2\Sigma mr^2$, k is called the radius of gyration, and is the distance at which a mass equal to that of the whole body must be concentrated to possess the same moment of inertia as the body possesses.

The formula for moment of inertia is, of course,

$$\Sigma mr^2, \tag{18}$$

and its dimensions are ML^2.

The moment of inertia of a body with reference to an axis passing through its centre of inertia being known, its moment of inertia with reference to any other axis parallel to this is found by adding to the moment of inertia already known the product of the mass of the body and the square of the distance of its centre of inertia from the new axis of rotation. For if the centre of inertia of the body whose moment of inertia we know be C, and if m be any particle of that body, and if O be the new axis to which the moment of inertia is to be referred, making the construction as in the figure (Fig. 21), we have

$$r^2 = r_1^2 + 2r_1b + b^2 + c^2,$$
$$a^2 = b^2 + c^2;$$

whence

$$r^2 = a^2 + 2r_1b + r_1^2.$$

Multiplying all through by the mass m, and performing a similar operation for every particle of the body, and summing the results, we have

$$I_1 = \Sigma ma^2 + \Sigma mr_1^2 + 2\Sigma mr_1 b.$$

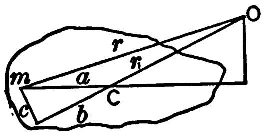

Fig. 21.

The term $2\Sigma mr_1 b$ on the right vanishes, for we may write it $2r_1\Sigma mb$; and, since C is the centre of inertia, $\Sigma(mb)$ is zero. Therefore

$$I_1 = I + Mr_1^2. \qquad (19)$$

Thus the proposition is proved.

The moment of inertia of the simple geometrical solids may be found by reckoning the moments of inertia for the separate particles of the body, and summing the results. We will show how this may be done in a few simple cases.

(1) To find the moment of inertia of a very thin rod AB, length $2l'$ and mass $2m'$, about an axis xx', passing through the middle point:—

Suppose the half-length to be divided into a very large number n of equal parts. The mass of each will be $\frac{m'}{n}$. The distance of the first from the axis is $\frac{l'}{n}$, of the second $\frac{2l'}{n}$, etc. Their moments of inertia are

$$\frac{m'}{n} \times \frac{l'^2}{n^2}, \quad \frac{m'}{n} \times 4\frac{l'^2}{n^2} \quad . \; . \; . \quad \frac{m'}{n} \times n^2\frac{l'^2}{n^2},$$

and the moment of inertia of the half-rod is

$$I' = \frac{m'l'^2}{n^3}(1 + 4 + 9 \ldots + n^2).$$

But $(1 + 4 + 9 \ldots + n^2)$, where n is very large, is $\frac{n^3}{3}$; hence $I' = \frac{m'l'^2}{3}$.

If l equals the whole length of the rod, m the whole mass, and I the entire moment of inertia,

$$I = \frac{ml^2}{12}. \qquad (20)$$

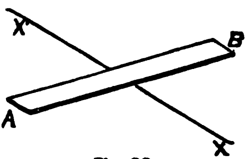

Fig. 22.

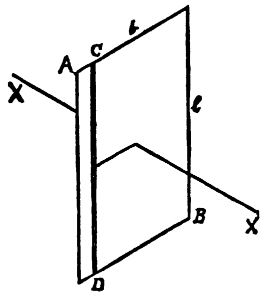

Fig. 23.

(2) To find the moment of inertia of a thin plate AB (Fig. 23), length l and breadth $2b'$, about an axis perpendicular to it and passing through its centre:—

Suppose the half-plate to be divided into n rods, parallel to its length: each rod will have a length l and a breadth $\frac{b'}{n}$. Their distances from the axis are $\frac{b'}{n}$, $\frac{2b'}{n}$, etc. Let m be the mass of the plate. The moment of inertia of each rod, with respect to an axis passing through its centre of inertia and perpendicular to its length, is $\frac{m}{2n} \times \frac{l^2}{12}$. The moments of inertia of the sev-

eral rods about the parallel axis xx' are

$$\frac{m}{2n}\left(\frac{l^2}{12} + \frac{b'^2}{n^2}\right),\ \frac{m}{2n}\left(\frac{l^2}{12} + 2\frac{b'^2}{n^2}\right), \text{ etc.,}$$

and the moment of inertia of the half-plate,

$$\frac{m}{2n} \times n\frac{l^2}{12} + \frac{m}{2n}\frac{b'^2}{n^2}(1 + 4 + 9 \ldots + n^2) = \frac{m}{2}\left(\frac{l^2}{12} + \frac{b'^2}{3}\right),$$

or, for the whole plate, equals

$$m\frac{l^2 + b^2}{12}. \qquad (21)$$

A parallelopiped whose axis is xx' may be supposed to be made up of an infinite number of plates, such as *AB*. Its moment of inertia will be the moment of inertia of one plate multiplied by the number of plates; or, if M is the mass of the parallelopiped, its moment of inertia is

$$\frac{M}{12}(l^2 + b^2). \qquad (22)$$

The moment of inertia of any body, however irregular in form or density, may be found experimentally by the aid of another body whose moment of inertia can be computed from its dimensions.

We will anticipate the law of the pendulum, which has not been proved, for the sake of clearness. The body whose moment of inertia is desired is set oscillating about an axis under the action of a constant force f. Its time of oscillation is, then,

$$t = \pi\sqrt{\frac{I}{f}},$$

where I is the moment of inertia.

If, now, another body, whose moment of inertia can be calculated, is joined with the first, the time of oscillation alters to

$$t_1 = \pi\sqrt{\frac{I + I_1}{f}},$$

where I_1 is the moment of inertia of the body added. Combining the two equations, we obtain, as the value of the moment of inertia desired,

$$I = \frac{I_1 t^2}{t_1^2 - t^2}. \qquad (23)$$

30. ***Central Forces.*** — If the velocity or direction of motion of a moving body in any way alter, we conceive it to be acted on by some force. In certain cases the direction of this force, and the law of its variation with the position of the body, may be determined by considering the path or orbit traversed by the body and the circumstances of its motion.

We shall illustrate this by a few propositions, selected on account of their applicability in the establishment of the "Theory of Universal Gravitation." The proofs are substantially those given by Newton in the "Principia."

Proposition I. — If the radius vector drawn from a fixed point to a body moving in a curve describe equal areas in equal times, the force which causes the body to move in the curve is directed towards the fixed point.

Let us suppose the whole time divided into equal periods, during any one of which the body is not acted on by the force. It will, in the first period, move over a space represented by a straight line, as AB (Fig. 24). In the second period, it would, if unhindered, move

over an equal space BD and in the same line. It, however, is deflected by a force acting instantaneously at the point B, and moves in a line BC such that, by hypothesis, triangle OBA = triangle OBC. Now, triangle ODB also = triangle OAB, therefore triangle OCB = triangle ODB, and CD is parallel to OB. Complete the parallelogram $CDBE$; then it is evident that the motion BC is compounded of the motions BD and BE; and since forces are proportional to the motions they occasion, the force acting at B is proportional to BE, and is directed along the line BO. If, now, the period of time becomes indefinitely small, the broken line ABC approaches indefinitely near to a curve, the force which causes the motion in the curve being always directed to the centre O.

Fig. 24.

Proposition II. — If a body move uniformly in a circle, the force acting upon it varies as its mass and the square of its velocity directly, and as the radius of the circle inversely.

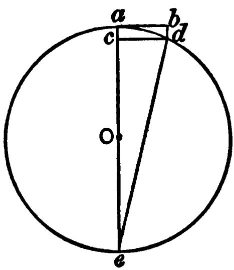

Fig. 25.

If the body m move in a circle (Fig. 25) with a constant angular velocity, and pass over, in any very small time t, the arc ad, which is so small that it may be taken equal to its chord, the motion may be resolved into two components ab and ac, one tangent and the other normal to the arc. Now f, the acceleration towards O, being constant for that small time, we have

$$s = ac = \tfrac{1}{2}ft^2.$$

By Euclid, iii. 31, *ade* is a right angle, and therefore, by similar triangles,

$$ae = \frac{ad^2}{ac}.$$

But $ae = 2r$; ad may be taken equal to ab, which represents the space traversed in the time t; and from the previous reasoning ac represents $\frac{1}{2}ft^2$: whence

$$\frac{ft^2}{2} = \frac{v^2t^2}{2r},$$

and

$$mf = \frac{mv^2}{r}.$$

Corollary I.—If two bodies are revolving about the same centre, and the squares of their periodic times are in the same ratio as the cubes of the radii of their respective orbits, the forces acting on them will be inversely as the squares of their radii, and conversely. For, if T and T_1 represent the periodic times of the two bodies moving in circles of radii r and r_1, with velocities v and v_1, then, by hypothesis,

$$T : T_1 = \frac{2\pi r}{v} : \frac{2\pi r_1}{v_1} = r^{\frac{3}{2}} : r_1^{\frac{3}{2}};$$

whence

$$v : v_1 = r_1^{\frac{1}{2}} : r^{\frac{1}{2}}.$$

Now

$$f : f_1 = \frac{v^2}{r} : \frac{v_1^2}{r_1};$$

whence

$$f : f_1 = r_1^2 : r^2.$$

Corollary II.—The relation of Corollary I. holds with reference to bodies describing similar parts of any simi-

lar figures having the same centre. In the application of the proof, however, we must substitute for uniform velocity the uniform description of areas; and instead of radii we must use the distances of the bodies from the centre. The proof is as follows:—

If D and D_1 represent the radii of curvature of the paths of the two bodies, R and R_1 the distances of the bodies from the centre of force, then, by hypothesis (letting A represent the area described in one period of time),

$$T : T_1 = \frac{R^2}{A} : \frac{R_1^2}{A_1} = R^{\frac{3}{2}} : R_1^{\frac{3}{2}} = D^{\frac{3}{2}} : D_1^{\frac{3}{2}},$$

from the similarity of figures.

Now

$$A : A_1 = vR : v_1R_1;$$

hence

$$v : v_1 = R_1^{\frac{1}{2}} : R^{\frac{1}{2}} = D_1^{\frac{1}{2}} : D^{\frac{1}{2}},$$

and

$$f : f_1 = \frac{v^2}{D} : \frac{v_1^2}{D_1} = R_1^2 : R^2.$$

Proposition III.—If a body move in an ellipse, the force acting upon it, directed to the focus of the ellipse, varies inversely as the square of the radius vector.

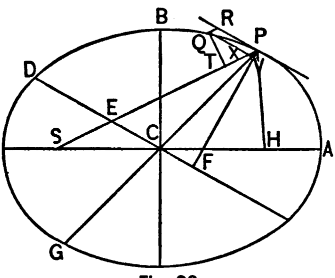

Fig. 26.

Suppose the body moving in the ellipse to be at the point P (Fig. 26), and the force to act upon it along the radius vector SP. At the point P draw the tangent PR, and from a point Q on the ellipse draw the

chord Qv, cutting SP in x, and complete the parallelogram $PRQx$. From Q draw QT perpendicular to SP. Also draw the diameter GP and its conjugate DK. The force which acts on the body, causing it to leave the tangent PR and move in the line PQ, acts along SP, and in a time t (supposed very small) causes the body to move in the direction SP over the space Px; and since, in the small time considered, it may be assumed constant,

$$Px = \tfrac{1}{2}ft^2;$$

whence

$$t^2 = \frac{2Px}{f}.$$

Again: the area described by the radius vector in the time t is equal to $\dfrac{SP \cdot QT}{2}$; and if A represent the area described in unit time,

$$At = \frac{SP \cdot QT}{2}.$$

Equating these values of t, we obtain

$$\frac{SP^2 \cdot QT^2}{4A^2} = \frac{2Px}{f};$$

whence

$$f = \frac{8A^2 \cdot Px}{QT^2} \cdot \frac{1}{SP^2}.$$

From Proposition I., the value of A is constant for any part of the ellipse. We shall now show that $\dfrac{Px}{QT^2}$ is also constant.

From similar triangles,

$$Px : Pv = PE : PC;$$

or, since by a property of the ellipse $PE = AC$,

$$Px : Pv = AC : PC.$$

Again, by another property of the ellipse,

$$Gv \cdot Pv : Qv^2 = PC^2 : CD^2.$$

If, now, we consider the time t to become indefinitely small in the limit, P and Q approach indefinitely near; whence $Qv = Qx$ and $Gv = 2PC$. The last proportion then becomes

$$PC \cdot Pv : Qx^2 = PC^2 : 2\,CD^2.$$

Again, from similar triangles,

$$Qx : QT = PE : PF = AC : PF;$$

and from another property of the ellipse,

$$AC : PF = CD : CB;$$

whence

$$Qx : QT = CD : CB.$$

Combining these proportions,

$$\begin{aligned} Px &: Pv &&= AC &&: PC, \\ Pv &: Qx^2 &&= PC &&: 2\,CD^2, \\ Qx^2 &: QT^2 &&= CD^2 &&: CB^2, \end{aligned}$$

we obtain, finally,

$$Px : QT^2 = AC : 2\,CB^2;$$

that is, since AC and CB are constant, $\dfrac{Px}{QT^2}$ is constant.

We have now shown, that, in the expression for the value of the force on the body at any point in the ellipse, all the factors are constant except $\dfrac{1}{SP^2}$. The force, therefore, varies inversely as the square of the radius vector.

CHAPTER II.

UNIVERSAL ATTRACTION.

31. ***Universal Attraction.***—The law of universal attraction was the first generalization of modern science. In its most complete form it may be stated as follows:—

Between every two material particles in the universe there is a stress, of the nature of an attraction, which varies directly as the product of the masses of the particles, and inversely as the square of the distance between them.

Some of the ancient philosophers had a vague belief in the existence of an attraction between the particles of matter. This hypothesis, however, with the knowledge which they possessed, was not susceptible of proof. The geocentric theory of the planetary system, which obtained almost universal acceptance, offered none of those beautiful and simple relations of the planetary motions upon which the law was finally established. It was not until the heliocentric theory, advocated by Copernicus, strengthened by the discoveries of Galileo, and systematized by the labors of Kepler, had been fully accepted, that the discovery of the law became possible.

In particular, the three laws of planetary motion published by Kepler in 1609 and 1619 laid the foundations for Newton's demonstrations. The laws are as follows:—

I. The planets move in ellipses of which one focus is situated at the sun.

II. The radius vector drawn from the sun to the planet sweeps out equal areas in equal times.

III. The squares of the periodic times of the planets are proportional to the cubes of their distances from the sun.

Kepler could give no physical reason for the existence of such laws. Later in the century, after Huyghens had discovered the theorems relating to centrifugal force, it was seen that the third law would hold true for bodies moving in concentric circles, and attracted to the common centre by forces varying inversely as the squares of the radii of the circles. Several English philosophers, among them Hooke, Wren, and Halley, based a belief in the existence of an attraction between the sun and the planets upon this theorem.

The demonstration was by no means a rigorous one, and was not generally accepted. It was left for Newton to show that not only the third, but all, of Kepler's laws were completely satisfied by the assumption of the existence of an attraction acting between the sun and the planets, and varying inversely as the square of the distance. His propositions are substantially given in the section on "Central Forces."

Newton next showed that the attraction holding the moon in its orbit, which is presumably of the same nature as that existing between the sun and the planets, is the same as that which causes heavy bodies to fall to the earth. This he accomplished by showing that the deviation of the moon from a rectilineal path is such as should occur if the force of gravity at the

earth's surface were to extend outwards to the moon, and vary in intensity inversely as the square of the distance.

Two further steps were necessary before the final generalization could be reached. One was, to show the relation of the attraction to the masses of the attracting bodies; the other, to show that this attraction exists between all particles of matter, and not merely, as Huyghens believed, between those particles and the centres of the sun and planets.

The first step was also taken by Newton. By means of pendulums having the same length, but with bobs of different materials, he showed that the force acting on a body at the earth's surface is proportional to the mass of the body, since all bodies have the same acceleration. He further brought forward, as the most satisfactory theory which he could form, the general statement that "every particle of matter attracts and is attracted by every other particle."

The experiments necessary for a complete verification of this last statement were not carried out by Newton. They were performed in 1798 by Cavendish. His apparatus consisted essentially of a bar furnished at both ends with small leaden balls, suspended horizontally by a long fine wire, so that it turned freely in the horizontal plane. Two large leaden balls were mounted on a bar of the same length, which rotated about a vertical axis coincident with the axis of rotation of the suspended bar. The large balls, therefore, could be set and clamped at any angular distance desired from the small balls. The whole arrangement was enclosed in a room, to prevent all disturbance. The

motion of the suspended system was observed from without by means of a telescope. Neglecting as unessential the special methods of observation employed, it is sufficient to state that an attraction was observed between the large and small balls, and was found to be in accordance with the law as above stated.

32. ***Measurement of the Force of Gravity.*** — When two bodies attract one another, their relative motions are determined by Newton's third law. In the case of the attraction between the earth and a body near its surface, the velocity acquired by the earth in a unit of time is so small that it may be neglected, and the acceleration of the body alone need be considered. Since the force acting upon it varies with its mass, and since its gain in momentum also varies with its mass, it follows that its acceleration will be constant, however its mass may vary. We may, therefore, obtain a direct measure of the earth's attraction, or of the "force of gravity," by allowing a body to fall freely *in vacuo*, and determining its rate of acceleration. It is found that a body so falling at latitude 40° will describe in one second about 16.08 feet, or 490 centimetres. Its acceleration is therefore 32.16 in feet and seconds, or 980 in centimetres and seconds. We denote this acceleration by the symbol g.

The force acting on the body, or the weight of the body, is seen at once to be mg, where m is the mass of the body.

On account of almost insuperable difficulties in the employment of this method, various others are used to obtain the value of g indirectly. For example, we may allow bodies to roll down a smooth inclined plane, and

observe their motion. The force effective in producing motion on the plane is $g \sin \phi$, where ϕ is the angle of the plane with the horizon; the space traversed in the time t, $s = \frac{1}{2}gt^2 \sin \phi$. By observing s and t, the value of g may be obtained. The motion is so much less rapid than that of a freely falling body that tolerably accurate observations can be made. Irregularities, due to friction upon the plane and the resistance of the air, however, greatly vitiate any calculations based upon these observations. This method was used by Galileo, who was the first to obtain a measure of the acceleration due to the earth's attraction.

The most exact method for determining the value of g is based upon observations of the oscillation of a pendulum.

A pendulum may be defined as a heavy mass, or bob, suspended from a rigid support, so that it can oscillate about its position of equilibrium.

In the simple, or mathematical, pendulum, the bob is assumed to be a material particle, and to be suspended by a thread without weight. If the bob be stationary and acted on by gravity alone, the line of the thread will be the direction of the force. If the bob be withdrawn from the position of equilibrium (Fig. 27), it will be acted on by a force at right angles to the thread, expressed by $g \sin \phi$, where ϕ is the angle between the perpendicular and the new position of the thread.

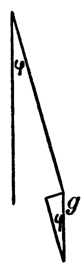

Fig. 27.

The force acting upon the bob at any point in the circle of which the thread is radius, if it be released, and allowed to swing in that circle, varies as the sine

of the arc cut off by the perpendicular and the radius drawn to that point. If we make the oscillation so small that the arc may be substituted for its sine without sensible error, the force acting on the bob varies as the displacement of the bob from the point of equilibrium. Now a body acted on by a force varying as the displacement of the body from a fixed point will have a simple harmonic motion about its position of equilibrium.

As a consequence, it may be noted that the oscillations of the pendulum are symmetrical about the position of equilibrium; and hence the bob will have an amplitude on the one side of the vertical equal to that which it has on the other, and the oscillations, once set up, will continue forever unless modified by outside forces.

The importance of the pendulum as a means of determining the value of g consists in this: that, instead of observing the space traversed by the bob in one second, we can observe the number of oscillations made in any period of time, and determine the time of one oscillation; from this, and the length of the pendulum, we can calculate the value of g. The errors in the necessary observations and measurements are very slight in comparison with those of any other method.

33. ***Formula for Simple Pendulum.***—The formula connecting the time of oscillation with the value of g is obtained as follows: The acceleration of the bob at any point in the arc is, as we have seen, $g \sin \phi$, or $g\phi$ if the arc be very small. The acceleration in a simple harmonic motion is

$$\pm \frac{4\pi^2}{T^2} a \cos \frac{2\pi t}{T}.$$

Since the bob has a simple harmonic motion, we may equate these expressions: hence

$$g\phi = \frac{4\pi^2}{T^2} a \cos\frac{2\pi t}{T}.$$

But $a \cos\frac{2\pi t}{T}$ is the displacement of the point having the simple harmonic motion, and is therefore equal to $l\phi$ if l represent the length of the thread: hence

$$g = \frac{4\pi^2 l}{T^2},$$

from which

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

In this formula T represents the time of a double oscillation. It is customary to consider as a unit the time of a single oscillation, when the formula becomes

$$t = \pi\sqrt{\frac{l}{g}}. \qquad (24)$$

34. ***Physical Pendulum.*** — Any pendulum fulfilling the requirements of the foregoing theory is, of course, unattainable in practice. We can, however, calculate, from the known dimensions and mass of the portions of matter making up the physical pendulum, what would be the length of a simple pendulum which would oscillate in the same time. It is clear that there must be some point in every physical pendulum whose distance from the point of suspension is equal to the length of the corresponding simple pendulum; for the particles nearer the point of suspension tend to oscillate more rapidly than those more remote, and the time of oscilla-

tion of the system, if rigid, will be intermediate between the extreme values. There will, therefore, be some one particle whose proper rate of oscillation is the same as that of the whole pendulum. Its distance from the point of suspension is the length sought.

(1) In determinations of the value of g by observations upon the time of oscillation of a pendulum, the length of the equivalent simple pendulum may be known in one of two ways. The pendulum may be constructed in such a manner that its moment of inertia and the position of its centre of gravity may be calculated. From these data the required length is readily obtained.

We have already shown that the moment of inertia of a body having a given angular velocity is proportional to its kinetic energy. The full formula is

$$E = \Sigma mr^2\frac{\omega^2}{2},$$

where r is the distance of any particle from the point taken as the centre, about which the angular velocity is reckoned. In the case of the pendulum, let us consider the velocity of any particle at the lowest point of the arc. It is, in terms of simple harmonic motion, $\frac{2\pi a}{T}$. The angular velocity of the system is, therefore, $\frac{2\pi\phi}{T}$, for $a = r\phi$. The kinetic energy of the pendulum at the lowest point of its arc is, therefore,

$$\tfrac{1}{2}\Sigma mr^2\frac{4\pi^2\phi^2}{T^2}.$$

This must be equal to the potential energy of the pendulum at the highest point of its arc; that is, to

$\frac{1}{2}MRg\phi^2$, where $\frac{1}{2}Mg\phi$ represents the average force acting on the centre of gravity, from the highest to the lowest point of the arc, to produce rotation; and, R being the distance of the centre of gravity from the point of suspension, $R\phi$ is the distance over which the average force acts. We therefore have

$$\frac{1}{2}\Sigma mr^2\frac{4\pi^2\phi^2}{T^2} = \frac{1}{2}MRg\phi^2;$$

whence

$$T = 2\pi\sqrt{\frac{\Sigma mr^2}{MR\cdot g}}. \qquad (25)$$

This is the time of oscillation of a simple pendulum whose length is $\frac{\Sigma mr^2}{MR}$. Therefore the moment of inertia of any physical pendulum divided by its static moment gives the length of the equivalent simple pendulum. The point on the line joining the point of suspension with the centre of gravity of the pendulum, and distant $\frac{\Sigma mr^2}{MR}$ from the point of suspension, is called the centre of oscillation.

A pendulum consisting of a heavy spherical bob suspended by a cylindrical wire was used by Borda in his determinations of the value of g. The moment of inertia and the centre of gravity of the system were easily calculated, and the length of the simple pendulum to which the system was equivalent was thus obtained.

(2) We may determine the length of the equivalent simple pendulum directly by observation. The method depends upon the principle, that, if the centre of oscillation be taken as the point of suspension, the time of

oscillation will not vary. The proof of this principle is as follows: —

Let r and $l - r$ represent the distances from the centre of gravity to the point of suspension and centre of oscillation respectively, m represent the mass of the pendulum, and I its moment of inertia about its centre of gravity. Then, since the moment of inertia about the point of suspension is $I + mr^2$, we have

$$l = \frac{I + mr^2}{mr}.$$

When the pendulum is reversed, we have

$$l_1 = \frac{I + m(l - r)^2}{m(l - r)}.$$

From the first equation we have $I = mr(l - r)$, which value substituted in the second gives, after reduction, $l_1 = l$; that is, the length of the equivalent simple pendulum, and consequently the time of oscillation when the pendulum swings about its point of suspension, is the same as that when it is reversed, and swings about its former centre of oscillation.

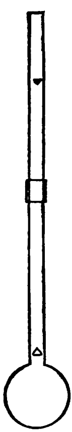

Fig. 28.

A pendulum (Fig. 28) so constructed as to take advantage of this principle was used by Capt. Kater in his determination of the value of g; and this form is known, in consequence, as Kater's pendulum.

35. ***The Balance.*** — The comparison of masses is of such frequent occurrence in physical investigations that it is important to consider the theory of the balance and the methods of using it.

To be of value, the balance must be accurate and sensitive; that is, it must be in the position of equilibrium when the scale-pans contain equal masses, and it must move out of that position on the addition to the mass in one pan of a very small fraction of the original load. These conditions are attained by the application of principles which have already been developed.

The balance consists essentially of a regularly formed beam, poised at the middle point of its length upon knife edges which rest on agate planes. From each end of the beam is hung a scale-pan, in which the masses

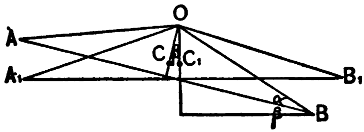

Fig. 29.

to be compared are placed. Let O (Fig. 29) be the point of suspension of the beam; A, B, the points of suspension of the scale-pans; C the centre of gravity of the beam, whose weight is W. Represent $OA = OB$ by l, OC by d, and the angle OAB by α.

If the weight in the scale-pan at A be P, and that in the one at B be $P + p$, where p is a small additional weight, the beam will turn out of its original horizontal position, and assume a new one. Let the angle COC_1, through which it turns, be designated by β. Then the moments of force about O are equal; that is,

$$(P + p)l \cdot \cos(\alpha + \beta) = Pl \cdot \cos(\alpha - \beta) + Wd \cdot \sin\beta;$$

from which we obtain, by expanding and transposing,

$$\tan\beta = \frac{pl\cos\alpha}{(2P + p)l\sin\alpha + Wd}. \qquad (26)$$

The conditions of greatest sensitiveness are readily deducible from this equation. As long as $\cos\alpha$ is less than unity, it is evident that $\tan\beta$, and therefore β, increases as the weight $2P$ of the load diminishes. As the angle α becomes less, the value of β also increases, until, when A, O, and B are in the same straight line, it depends only on $\frac{pl}{Wd}$, and is independent of the load. In this case $\tan\beta$ increases as d, the distance from the point of suspension to the centre of gravity of the beam diminishes, and as the weight of the beam W diminishes. To secure sensitiveness, therefore, the beam must be as long and as light as is consistent with stiffness, the points of suspension of the beam and of the scale-pans must be very nearly in the same line, and the distance of the centre of gravity from the point of suspension of the beam must be as small as possible. Great length of beam, and near coincidence of the centre of gravity with the axis, are, however, inconsistent with rapidity of action. The purpose for which the balance is to be used must determine, therefore, the extent to which these conditions of sensitiveness shall be carried.

Accuracy is secured by making the arms of the beam of equal length, and so that they will perfectly balance, and by attaching scale-pans of equal weight at equal distances from the centre of the beam.

In the balances usually employed in physical and chemical investigations, various adjustments are provided, by means of which all the required conditions may be secured. The beam is poised on knife edges; and the adjustment of its centre of gravity is made by

changing the position of a nut which moves on a screw, placed vertically, directly above the point of suspension. Perfect equality in the moments of force due to the two arms of the beam is secured by a similar horizontal screw and nut placed at one end of the beam. The beam is a flat rhombus of brass, large portions of which are cut out so as to make it as light as possible. The knife edge on which the beam rests, and those upon which the scale-pans hang, are arranged so that, with a medium load, they are all in the same line. A long pointer attached to the beam moves before a scale, and serves to indicate the deviation of the beam from the position of equilibrium. If the balance be accurately made and perfectly adjusted, and equal weights placed in the scale-pans, the pointer will remain at rest, or will oscillate through distances regularly diminishing on each side of the zero of the scale.

If the weight of a body is to be determined, it is placed in one scale-pan, and known weights are placed in the other until the balance is in equilibrium or nearly so. The final determination of the exact weight of the body is then made by one of three methods: we may continue to add very small weights until equilibrium is established; or we may observe the deviation of the pointer from the zero of the scale, and, by a table prepared empirically, determine the excess of one weight over the other; or we may place a known weight at such a point on a graduated bar attached to the beam that equilibrium is established, and find what its value is, in terms of weight placed in the scale-pan, by the relation between the length of the arm of the beam and the distance of the weight from the middle point of the beam.

If the balance be not accurately constructed, we can, nevertheless, obtain an accurate value of the weight desired. The method employed is known as Borda's method of double weighing. The body to be weighed is placed in one scale-pan, and balanced with fine shot or sand placed in the other. It is then replaced by known weights till equilibrium is again established. It is manifest that the replacing weights represent the weight of the body.

If the error of the balance consist in the unequal length of the arms of the beam, the true weight of a body may be obtained by weighing it first in one scale-pan and then in the other. The geometrical mean of the two values is the true weight; for let l_1 and l_2 represent the lengths of the two arms of the balance, P the true weight, and P_1 and P_2 the values of the weights placed in the pans at the extremities of the arms of lengths l_1 and l_2, which balance it. Then $Pl_2 = P_1 l_1$, and $Pl_1 = P_2 l_2$; from which

$$P = \sqrt{P_1 P_2}.$$

36. ***Density of the Earth.*** — One of the most interesting problems connected with the physical aspect of gravitation is the determination of the density of the earth. It has been attacked in several ways, each of which is worthy of consideration.

The first successful determination of the earth's density was based upon experiments made in 1774 by Maskelyne. He observed the deflection from the vertical of a plumb-line suspended near the mountain Schehallien in Scotland. He then determined the density of the mountain by the specific gravity of specimens of earth

and rock from various parts of it, and calculated the ratio of the volume of the mountain to that of the earth. From these data the mean specific gravity of the earth was determined to be about 4.5.

The next results were obtained from the experiments of Cavendish, in 1798, with the torsion balance already described. The density, volume, and attraction of the leaden balls being known, the density of the earth could easily be obtained. Cavendish's value was about 5.5.

Another method, employed by Carlini in 1824, depends upon the use of the pendulum. The time of the oscillation of a pendulum at the sea-level being known, the pendulum is carried to the top of some high mountain, and its time of oscillation again observed. The value of g as deduced from this observation will, of course, be less than that obtained by the observation at the sea-level. It will not, however, be as much less as the law of inverse squares would indicate. The discrepancy is due to the attraction of the mountain, which can, therefore, be calculated, and the calculations completed, as in Maskelyne's experiment.

A fourth method, due to Sir G. B. Airy, and employed by him in 1854, consists in observing the time of oscillation of a pendulum at the bottom of a deep mine. By Proposition I., § 22, it appears that the attraction of a spherical shell of earth whose thickness is the depth of the mine vanishes. The mean density of the earth may, therefore, be determined by the discrepancy between the values of g at the bottom of the mine and at the surface.

These methods have yielded results varying from that of Airy, who stated the mean specific gravity to

be between 6 and 7, to that of Maskelyne, who obtained 4.5. The most elaborate experiments, by MM. Cornu and Baille, by the method of Cavendish, gave as the value 5.56. This is probably very near the truth.

37. ***Projectiles.*** — When a body is projected in any direction near the earth's surface, it follows, in general, a curved path. If the lines of force be considered as radiating from the earth's centre, this path will be, by Proposition III., § 30, an ellipse, with one focus at the earth's centre. If the path pursued be so small that the lines may be considered parallel, the centre of force is conceived of as removed to an infinite distance, and the curve becomes a parabola.

The fact that ordinary projectiles follow a parabolic path was first shown by Galileo, as a deduction from the principle which he established, — that a constant force produces a uniform acceleration. The proof is as follows: Suppose the body to be projected in the direction OY (Fig. 30), making any angle ϕ with OX, a vertical line, and to move with a velocity $v = \frac{s}{t}$. Owing to the accelerating effect of gravity, it also moves in the vertical direction OX with a velocity $v_1 = gt$. At any time t it will have traversed in the direction OY a space $y = vt$, and in the direction OX a space $x = \frac{1}{2}gt^2$. The co-ordinates of the position of the body at any time t are, therefore, $y = vt$ and $x = \frac{1}{2}gt^2$, if OY and OX are the axes. The equation connecting x and y becomes $y^2 = \frac{2v^2x}{g}$, which is the equation of a parabola referred to the diameter

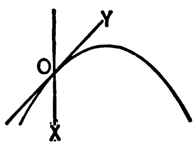

Fig. 30.

OX and the tangent OY. When the body is projected horizontally, the vertex of the parabola is at the origin of the motion. The body begins to approach the earth from the start, and reaches it in the same time that it would if allowed to fall freely.

One special case of importance in the consideration of the paths of projectiles is that in which the body moves in a circle. It is obvious, that, to bring about this result, the body must be projected horizontally with such an initial velocity that the acceleration due to the earth's attraction shall be precisely equal to the acceleration toward the centre which is necessary in order that the body should move in a circle; that is,

$$\frac{mv^2}{R} = \frac{mM}{R^2}K,$$

where m and M are the masses of the body and the earth respectively, R is the earth's radius, and K the constant of attraction,—that is, the value of the stress exerted between two unit masses at unit distance apart. Now v, the velocity of the body, equals

$$\frac{2\pi R}{T},$$

where T is the time of one complete revolution, and

$$M = \tfrac{4}{3}\pi R^3 D,$$

where D is the earth's mean density. Substituting these values, we obtain

$$\frac{4m\pi^2R^2}{T^2R} = \frac{4m\pi R^3D}{3R^2}K,$$

from which

$$T^2 = \frac{3\pi}{DK}.$$

The result shows that the periodic time of any small body revolving about a sphere, and infinitely near its surface, is a function of the density only, and does not depend on the radius of the sphere.

Upon this principle Prof. J. Clerk Maxwell proposed, as an absolute unit of time, the time of revolution of a small satellite revolving infinitely near the surface of a globe of pure water at its maximum density.

CHAPTER III.

MOLECULAR MECHANICS.

CONSTITUTION OF MATTER.

38. ***General Properties of Matter.***—Besides the properties already defined in the introduction as characteristic and essential, we find that all bodies possess the properties of compressibility and divisibility.

Compressibility.—All bodies change in volume by change of pressure and temperature. If a body of a given volume be subjected to pressure, it will return to its original volume when the pressure is removed, provided the pressure has not been too great. This property of assuming its original volume is called elasticity. The property of changing volume by the application of heat is sometimes specially called dilatability.

Divisibility.—Any body of sensible magnitude may, by mechanical means, be divided, and each of its parts may again be subdivided; and the process may be continued till the resulting particles become so minute that we are no longer able to recognize them, even when assisted by the most perfect appliances of the microscope. If the body be one that can be dissolved, it may be put in solution, and this may be greatly diluted; and in some cases the body may be detected by the color which it imparts to the diluent, even when

constituting so small a proportion as $\frac{1}{1000000000}$ part of the solution (Hofmann).

39. ***Molecules.*** — We are not, however, at liberty to conclude that matter is infinitely divisible. The fact, established by observation, that bodies are impenetrable, and the one just noted, that they are at the same time compressible, as well as other considerations, to be adduced later, lead to the opposite conclusion. To explain the co-existence of these properties, we are compelled to assume that bodies are composed of extremely small portions of matter, indivisible without destroying their identity, called *molecules*, and that these molecules are separated by interstitial spaces relatively larger, which are occupied by a highly elastic medium called the *ether*.

These molecules can be divided only by chemical means. The resulting subdivisions are called atoms, which, however, cannot exist in a free state. The molecule is the physical unit of matter, while the atom is the chemical unit.

40. ***Composition of Bodies.*** — We have just seen that atoms cannot exist in a free state. They may be combined with others of the same kind or with those of dissimilar kind, whence we have either simple or compound substances.

There are about sixty-seven substances now known which cannot, in the present state of our knowledge, be decomposed, or made to yield any thing simpler than themselves. We therefore call them simple substances, elements, or, if we desire to avoid expressing any theory concerning them, we call them radicals. It is not improbable that some of these will yet be divided,

perhaps all of them. We can call them elements, then, only provisionally.

41. ***States of Aggregation.***—Bodies exist in three states,—the solid, the liquid, and the gaseous. In the solid state the form and volume of the body are both definite. In the liquid state the volume only is definite, the liquid assuming the form of the containing vessel. In the gaseous state neither form nor volume are determinate; since a gas can expand, and fill an indefinite space.

Many substances may, under proper conditions, assume either of these three states of aggregation; and some substances, as, for example, water, may exist in the three states under the same general conditions.

It is proper to add, however, that there is no such sharp line of distinction between the three states of matter as our definitions imply. Bodies present all gradations of aggregation between the extreme conditions of solid and gas; and the same substance, in passing from one state to the other, often presents all these gradations.

42. ***Structure of Solids.***—With the exception of organized bodies, all solids may be divided into two classes. The bodies of one class are characterized by more or less regularity of form, which is called crystalline; those of the other class, exhibiting no such regularity, are called amorphous. For the formation of crystals a certain amount of freedom of motion of the molecules is necessary. Such freedom of motion is found in the gaseous and liquid states; and when crystallizable bodies pass slowly from these to the solid state, crystallization usually occurs. It may also occur

in some solids spontaneously, or in consequence of agitation of the molecules by mechanical means, as friction, percussion, etc.

Some amorphous bodies cannot, under any circumstances, assume the crystalline form. They are called colloids.

43. ***Crystal Systems.*** — Crystals are arranged by mineralogists in six systems.

In the first, or *Isometric*, system, all the forms are referred to three equal axes at right angles. The system includes the cube, the regular octahedron, and the rhombic dodecahedron.

In the second, or *Dimetric*, system, all the forms are referred to a system of three rectangular axes, of which only two are equal.

In the third, or *Hexagonal*, system, the forms are referred to four axes, of which three are equal, lie in one plane, and cross each other at angles of 60°. The fourth axis is at right angles to the plane of the other three, and passes through their common intersection.

The fourth, or *Orthorhombic*, system, is characterized by three rectangular axes of unequal length.

In the fifth, or *Monoclinic*, system, the three axes are unequal. One of them is at right angles to the plane of the other two. The angles which these two make with each other, as well as the relative lengths of the axes, vary greatly for different substances.

In the sixth, or *Triclinic*, system, the three axes are oblique to each other, and unequal in length.

44. ***Forces determining the Structure of Bodies.*** — In view of what precedes, it is necessary to assume the existence of certain forces acting between the molecules

of matter. In liquids and solids, there must be a force in the nature of attraction, holding the molecules together, and a force equivalent to repulsion, preventing actual contact. The attractive force is called cohesion when it unites molecules of the same kind, and adhesion when it unites molecules of different kinds. The repulsive force is probably a manifestation of that motion of the molecules which constitutes heat. In gases this motion is so great as to carry the molecules beyond the limit of their mutual attractions: thus the apparent repulsion prevails, and the gas only ceases expanding when this repulsion is balanced by outside forces.

45. ***Structure of the Molecule.*** — The facts brought to light in the study of crystals compel us to ascribe a structural form to the molecule, determining special points of application for the molecular forces. From this results the arrangement of molecules having the requisite freedom of motion, into regular crystalline forms.

CAPILLARY PHENOMENA.

46. ***Fundamental Facts.*** — If we immerse one end of a fine glass tube having a very small (or capillary) bore in water, we observe that the water rises in the tube above its general level. We also observe that it rises around the outside of the tube, so that its surface in the immediate vicinity of the tube is curved. If we immerse the same tube in mercury, the surface of the mercury within and just outside the tube, instead of being elevated, is depressed. If we change the tube for one of smaller bore, the water rises higher and the mercury sinks lower within it; but the rise or depres-

sion outside the tube remains the same. If we immerse the same tube in different liquids, we find that the heights to which they ascend vary for the different liquids. If, instead of changing the diameter, we change the thickness of the wall, of the tube, no variation occurs in the amount of elevation or depression; and, finally, the rise or depression in the tube varies for any one liquid with its temperature.

47. ***Law of Force assumed.***—It is found that the ordinary force of gravitation following the law of inverse squares is not sufficient to produce these phenomena. They can, however, be explained if we assume an additional attraction between the molecules, such as we have already done. The analytical expression, then, of the stress between two molecules m and m' at distance r is

$$F = \frac{mm'}{r^2} + mm'f(r).$$

The only law which it is necessary to assign to the function of r in the second term is, that it is very great at insensible distances, diminishes rapidly as r increases, and vanishes while r, though measurable, is still a very small quantity. For adjacent molecules this molecular attraction is so much greater than the ordinary gravitational attraction, that it is customary, in the discussion of capillary phenomena, to omit the term $\frac{mm'}{r^2}$ from the expression for the force. The distance through which this attraction is appreciable is often called the radius of molecular action, and is denoted by the symbol ϵ. It is a very small distance, but is much greater than the distance between adjacent molecules.

48. ***Methods of Development.*** — The different methods which have been employed to deduce, from this assumed attraction, results which could be submitted to experimental verification, are worthy of notice. They are distinct, though compatible with one another. Thomas Young was the first to treat the subject satisfactorily, though others had given partial and imperfect demonstrations before him. He showed that a liquid can be dealt with as if it were covered at the bounding surface with a stretched membrane, in which is a constant tension tending to contract it. From this basis he proceeded to deduce some of the most important of the experimental laws. Laplace, proceeding directly from the law of the attraction which we have already given, considered the attraction of a mass of liquid on a filament of the liquid terminating at the surface, and obtained an expression for the pressure within the mass at the interior end of the filament. He also was able, not only to account for already observed laws, but to predict, in at least one instance, a subsequently verified result. Some years later, Gauss, dissatisfied with Laplace's assumption, without *a priori* demonstration of a known experimental fact, treated the subject from the basis of the doctrine of virtual velocities, which in this case is the equivalent of that of the conservation of energy. He proved, that, if any change be made in the form of a liquid mass, the work done or the energy recovered is proportional to the change of surface, and hence deduced a proof of the fact which Laplace assumed, and also an expression for the pressure within the mass of a liquid identical with his. For purposes of elementary treatment the earliest method is still the

best. We shall accordingly employ the idea of surface tension, after having shown that it may be obtained from our first hypothesis.

49. ***Surface Tension.*** — Let us consider any liquid bounded by a plane surface, of which the line mn (Fig. 31) is the trace, and let the line $m'n'$ be the trace of a parallel plane whose distance from the plane of mn is ϵ. The liquid is then divided into two parts by the plane of $m'n'$, — the general mass of the liquid, and a shell of thickness ϵ between the two planes. Then, if we imagine a plane passed through any point within the mass, it is clear that the attraction of the molecules

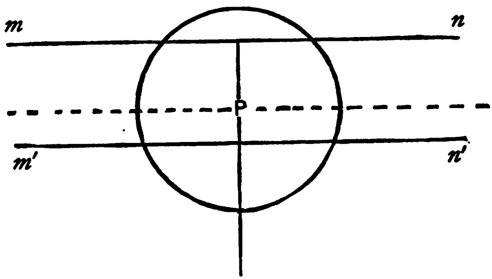

Fig. 31.

on opposite sides of that plane will give rise to a pressure normal to it, which will be constant for every direction of the plane; for the number of molecules now acting on the point is the same in all directions. If, however, the point chosen be P, situated within the shell, the pressure on the plane through P parallel to mn will be less than that on the plane through P normal to mn. With P as a centre, and with radius ϵ, describe a sphere. Now, it is evident that the number of molecules active in producing pressure upon the par-

allel plane is less than those producing pressure upon the normal plane. The pressure upon the parallel plane varies as we pass from the mass through the shell, from the value which it has within the mass, to 0, which it has at the plane *mn*. From this inequality of pressure in the two directions parallel and normal to the surface, there results a stress or tension of the nature of a contraction in the surface.

Provided the radius of curvature of the surface be not very small, this tension will be constant for the surface of each liquid, or, more properly, for the surface of separation between two liquids.

50. ***Energy and Surface Tension.*** — We may here show how the energy of the liquid is related to the surface tension. It is plain, that, if the molecules, whose mutual attractions give rise to the surface tension, be forced apart by the extension from the mass into the shell of a sheet of molecules along a plane normal to the surface, work will be done as the surface is increased. In every system free to move, movements will occur until the potential energy becomes a minimum: hence every free liquid moves so that its bounding-surface becomes as small as possible; that is, it assumes a spherical form. This is exemplified in falling drops of water and in globules of mercury, and can be shown on a large scale by a method soon to be described. If we call the potential energy lost by a diminution in the surface of one unit, the surface energy per unit surface, we can show that it is numerically equal to the surface tension across one unit of length.

Suppose a thin film of liquid to be stretched on a frame *ABCD* (Fig. 32), of which the part *BCD* is solid

and fixed, and the part A is a light rod, free to slide along C and D. The tendency of this film is, as we have said, to diminish its free surface. As it contracts, it draws A towards B. If the length of A be a, and A be drawn towards B over b units, then if E represent the surface energy per unit of surface, the energy lost or the work done is expressed by Eab. If we consider the tension acting normal to A, whose value is T for

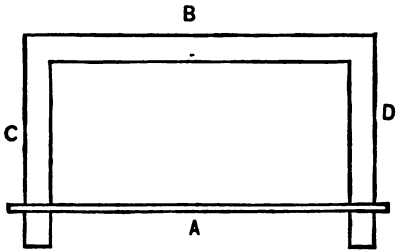

Fig. 32.

every unit of length, we have again for the work done during the movement of A, Tab. From these expressions we obtain at once $E = T$; that is, the numerical value of the surface energy per unit of surface is equal to that of the tension in the surface normal to any line in it per unit of length of that line.

51. ***Equation of Capillarity.***—The surface tension introduces modifications in the pressure within the liquid mass, depending upon the curvature of the surface. Consider any infinitesimal rectangle (Fig. 33) on the surface, the length of whose sides is s and s_1 respectively, and let R and R_1 be the radii of curvature of those sides. Also let ϕ and ϕ_1 be the angles in circular measures subtended by the sides from their respective centres of curvature. Now, a tension T for every unit

of length acts normal to s and tangent to the surface. The total tension across s is then Ts; and if this tension be resolved parallel and normal to the normal at the point P, the centre of the rectangle, we obtain for the parallel component $Ts \sin \frac{\phi_1}{2}$, or, since ϕ_1 is a very small angle, $Ts \frac{\phi_1}{2}$ or $Ts \frac{s_1}{2R_1}$. The opposite side gives a similar component; the side s_1 and the opposite give each a component $Ts_1 \frac{s}{2R}$. The total force along the normal at P is then

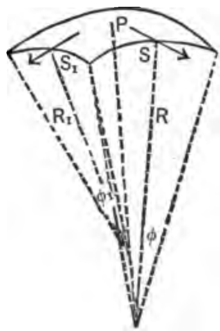

Fig. 33.

$$Tss_1\left(\frac{1}{R_1} + \frac{1}{R}\right);$$

and since ss_1 is the area of the infinitesimal rectangle, the force or pressure normal to the surface at P referred to unit of surface is

$$T\left(\frac{1}{R_1} + \frac{1}{R}\right).$$

From a theorem of Euler's we know that the sum $\frac{1}{R_1} + \frac{1}{R}$ is constant at any point for any position of the rectangular normal plane sections: hence the expression we have obtained fully represents the pressure at P.

If the surface is convex, the radii of curvature are positive, and the pressure is directed towards the liquid; if concave, they are negative, and the pressure is directed outwards. This pressure is to be added to the

constant molecular pressure which we have already seen exists everywhere in the mass. If we denote this constant molecular pressure by K, the expression for the total pressure within the mass is

$$K + T\left(\frac{1}{R_1} + \frac{1}{R}\right),$$

where the convention with regard to the signs of R_1 and R must be understood. For a plane surface, the radii of curvature are infinite, and the pressure under such a surface reduces to K.

52. ***Angles of Contact.*** — Many of the capillary phenomena appear when different fluids, or fluids and solids, are brought in contact with one another. It becomes, therefore, necessary to know the relations of the surface tensions and the angles of contact. They are determined by the following considerations: —

Consider first the case when three fluids meet along a line. Let O represent the point where this line cuts a plane drawn at right angles to it. Then the tension T_{ab} of the surface of separation of the fluid a from the fluid b, acting normal to this line, is counterbalanced by the tensions T_{ac} and T_{bc} of the surfaces of separation of a and c, b and c. These tensions are always the same for the three liquids under similar conditions of temperature and purity. Knowing the value of the tensions, the angles which they make with one another are determined at once by the parallelogram law; and these angles are always constant.

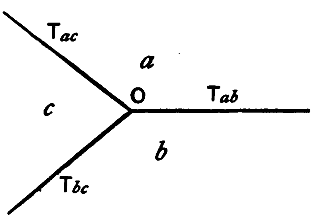

Fig. 34.

If T_{ab} is greater than the sum of T_{ac} and T_{bc}, the angle between T_{ac} and T_{bc} becomes zero, and the fluid C spreads itself out in a thin sheet between a and b. Thus, if a drop of oil be placed on water, the tension of the surface of separation between the air and water is greater than the sum of the tensions of the surfaces between the air and oil, and between the oil and water: hence the drop of oil spreads out over the water until it becomes almost indefinitely thin.

In the case of two fluids in contact with a plane solid (Fig. 35), it is evident that when the system is in equilibrium, the surface of separation between the fluids a and b, making the angle θ with the solid c,

$$T_{ac} = T_{bc} + T_{ab} \cos \theta.$$

The angle of contact is then determined by the equation

$$\cos \theta = \frac{T_{ac} - T_{bc}}{T_{ab}}.$$

If T_{ac} be greater than $T_{ab} + T_{bc}$, the equation gives an impossible value for $\cos \theta$. In this case the angle becomes evanescent, the fluid b spreads itself out, and wets the whole surface of the solid. In other cases the value of θ is finite and constant for the same substances. Thus, a drop of water placed on a horizontal glass plate will spread itself over the whole plate; while a small quantity of mercury placed on the same plate will gather together into a drop, whose edges make a constant angle with the surface.

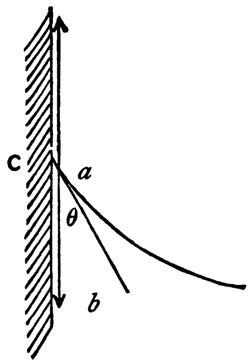

Fig. 35.

53. ***Plateau's Experiments.*** — The preceding principles will enable us to deduce a few of the most important experimental facts of capillarity.

A very interesting series of results was obtained by Plateau from the examination of the behavior of a mass of liquid removed from the action of gravity. His method of procedure was to place a mass of oil in a mixture of alcohol and water, carefully adjusted, to have the same specific gravity as the oil. The oil then had no tendency to move as a mass, and was free to arrange itself entirely under the action of the molecular forces. Referring to the equation of Laplace, already obtained, it is evident that equilibrium can exist only when the sum $\left(\frac{1}{R_1} + \frac{1}{R}\right)$ is constant for every point on the surface. This is manifestly a property of the sphere, and is true of no other finite surface. Plateau found, accordingly, that the freely floating mass at once assumed a spherical form. This result we had previously reached by another method. If a solid body — for instance, a wire frame — be introduced into the mass of oil of such a size as to reach the surface, the oil clings to it, and there is a break in the continuity of the surface at the points of contact. Each of the portions of the surface divided from the others by the solid then takes a form which fulfils the condition already laid down, that $\left(\frac{1}{R_1} + \frac{1}{R}\right)$ equals a constant. Plateau immersed a wire ring in the mass of oil. So long as the ring nowhere reached the surface, the mass remained spherical. On withdrawing a portion of the oil with a syringe, that which was left took the form of two equal *calottes*, or

sections of spheres, forming a double convex lens. A mass of oil, filling a short, wide tube, projected from it at either end in a similar section of a sphere. As the oil was removed, the two end surfaces became less curved, then plane, and finally concave.

Plateau also obtained portions of other figures which fulfil the requisite condition. For example, a mass of oil was made to surround two rings placed at a short distance from one another. Portions of the oil were then gradually withdrawn, when two spherical *calottes* formed, one at each ring, and the mass between the rings became a right cylinder. It is evident that the cylinder fulfils the required condition for every point on its surface.

Plateau also studied the behavior of films. He devised a mixture of soap and glycerine, which formed very tough and durable films; and he experimented with them in air. Such films are so light that the action of gravity on them may be neglected in comparison with that of the surface tension. If the parts of the frame upon which the film is stretched be all in one plane, the film will manifestly lie in that plane. When, however, the frame is constructed so that its parts mark the edges of any geometrical solid, the films which are taken up by it often meet. Any three films thus meeting so arrange themselves as to make angles of 120° with one another. This follows as a consequence of the proposition which has already been given to determine the equilibrium of surfaces of separation meeting along a line. If four or more meet, they always meet at a point.

Plateau also measured the pressure of air in a soap-

bubble, and found that it differed from the external pressure by an amount which varied inversely as the radius of the bubble. This follows at once from Laplace's equation. This measurement also gives us a means of determining the surface tension; for, from Laplace's equation, the pressure inwards, due to the outer surface, is $T \cdot \frac{2}{R}$, and the pressure in the same direction due to the inner surface is also $T \cdot \frac{2}{R}$, for the film is so thin that we may neglect the difference in the radii of curvature of the two surfaces: hence the total pressure inwards is $\frac{4T}{R}$; and if this be measured by a manometer, we can obtain the value of T.

54. ***Liquids influenced by Gravity.*** — Passing now to consider liquid masses acted on by gravity, we shall treat only a few of the most important cases.

JURIN'S LAW. — We notice first, that, if a glass tube having a narrow bore be immersed perpendicularly in water, the water rises in the tube to a height inversely proportional to the diameter of the tube. This law is known as Jurin's law.

Let Fig. 36 represent the section of a tube of radius r immersed in a liquid, the surface of which makes an angle θ with the wall. Then if T is the surface tension of the liquid, the tension acting upward is the component of this surface tension parallel to the wall, exerted all around the circumference of the tube. This is expressed by $2\pi r T \cos\theta$. This force, for each unit area of the tube, is

$$\frac{2\pi r T \cos\theta}{\pi r^2}.$$

The downward force, at the level of the free surface, making equilibrium with this, is due to the weight of the liquid column. If we neglect the weight of the meniscus, this force per unit area is expressed by hdg,

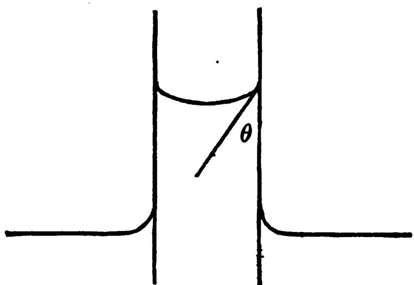

Fig. 36.

where h is the height of the column and d the density of the liquid. We have, accordingly,

$$\frac{2\pi r}{\pi r^2} T\cos\theta = hdg;$$

whence

$$h = \frac{2T\cos\theta}{rdg},$$

and the height is inversely as the radius of the tube.

If the liquid rises between two parallel plates of length l at distance r, the upward tension is determined by the expression $\frac{2l}{lr} T\cos\theta$, and the downward pressure by hdg; whence

$$h = \frac{2T\cos\theta}{rdg},$$

and the height to which the liquid rises between two

such plates is equal to that to which it rises in a tube whose radius is equal to the distance between the plates.

If the two plates are inclined to one another so as to touch along one vertical edge, the elevated surface takes the form of a rectangular hyperbola; for let the line of contact of the plates be taken as the axis of ordinates, and a line drawn in the plane of the free surface of the liquid as the axis of abscissæ, the elevations corresponding to each abscissa are inversely as the distance between the plates at that point, and are therefore inversely as the abscissæ: hence the product of any abscissa by its corresponding ordinate is a constant. The extremities of the ordinates then mark out a rectangular hyperbola referred to its asymptotes.

55. ***Liquid Drops in Capillary Tubes.***—When a drop of liquid is placed in a conical tube, it moves, if the

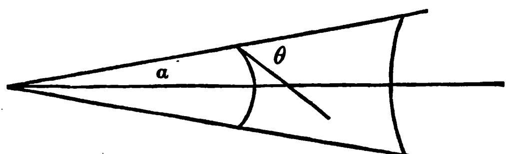

Fig. 37.

surfaces are concave, towards the smaller end; if convex, towards the larger end. The explanation of these movements follows readily from the foregoing results. In case the surfaces are concave, letting θ (Fig. 37) be the angle of contact and a the angle of inclination of the wall of the tube to the axis, r and r_1 the radii of the tube at the extremities of the drop, r being the smaller of the

two, then the expressions for the tensions acting in both cases outwards are respectively

$$\frac{2\pi r T}{\pi r^2}\cos(\theta - \alpha)$$

and

$$\frac{2\pi r_1 T}{\pi r_1^2}\cos(\theta + \alpha).$$

Of these two expressions the former is manifestly greater than the latter: hence the tendency of the drop is to move towards the smaller end of the tube.

If we assume that the concave surfaces are portions of spheres, of which R and R_1 are the respective radii of curvature, it follows very simply that $r = R\cos(\theta - \alpha)$, and $r_1 = R_1\cos(\theta + \alpha)$: hence the expressions for the tensions become $\frac{2T}{R}$ and $\frac{2T}{R_1}$. These are the values of the tensions as determined by Laplace's equation, and the movements of the drop might have been inferred directly from this equation by making the same assumption.

If a drop of water is introduced into a cylindrical capillary tube of glass, and if the air on the two ends of the drop has unequal pressures, the concavities thereby become unequal, the one on the side of the greater pressure presenting the greater concavity. The bead so circumstanced offers a resistance to this pressure; and it may, if the pressure be not too great, entirely counterbalance it. It is also evident, that, if several such beads are introduced successively, with intervening air-spaces, the pressure which they can unitedly sustain is equal to that which one can sustain multiplied by their number. Jamin found, that, with a tube containing a large number of beads, a pressure

of three atmospheres was maintained without diminution for fifteen days.

56. ***Movements of Solids.***—In certain cases the action of the capillary forces produces movements in solid bodies partially immersed in a liquid. For example, if two plates, which are both either wetted or not wetted by the liquid, are partially immersed vertically, and brought so near together that the rise or depression of the liquid due to the capillary action begins, then the plates will move towards one another. In either case this movement is explained by the inequality of pressure on the two sides of each plate. When the liquid rises between the plates, the pressure is zero at that point in the column which lies in the same plane as the free external surface. At every internal point above this the molecules of the liquid are in a state of negative pressure or tension, and the plates are consequently drawn together. When the liquid is depressed between the plates, they are pressed together by the external liquid above the plane in which the top of the column between the plates lies. When one of the plates is wetted by the liquid and the other not, the plates move apart. This is explained by noting, that, if the plates are brought near together, the convex surface at the one will meet the concave surface at the other, and there will be a consequent diminution in both the elevation and the depression at the inner surfaces of the plates. The elevation and depression at the outer surfaces remaining unchanged, there will result a pull outwards on the wetted plate and a pressure outwards on the plate which is not wetted; and they will consequently move apart. Laplace showed, however, as the

result of an extended discussion, that, though seeming repulsion exists between two plates such as we have just considered, yet, if the distance between the plates is diminished, this repulsion changes to an attraction. This prediction has been completely verified by the most careful experiments.

57. ***Porous Bodies.*** — Porous bodies may be considered as assemblages of more or less irregular capillary tubes. Thus the explanation of many natural phenomena — as the wetting of a sponge, the rise of the oil in the wick of a lamp — follows directly from the preceding discussion.

DIFFUSION OF LIQUIDS.

58. ***Free Diffusion.*** — When two liquids which are miscible are so brought together in a common vessel that the heavier is at the bottom and the lighter rests upon it in a well-defined layer, it is found that after a time, even though no agitation occur, they become uniformly mixed. Molecules of the heavier liquid make their way upwards through the lighter; while those of the lighter make their way downwards through the heavier, in apparent opposition to gravitation. Diffusion is the name which is employed to designate this phenomenon and others of a similar nature.

If one of the liquids is colored, — as, for example, solution of cupric sulphate, — while the other is colorless, the progress of the experiment may easily be watched and noted. If both liquids are colorless, small glass spheres, adjusted and sealed so as to have different but determined specific gravities between those of the liquids employed, may be placed in the vessel used in

the experiment, when they will show by their positions the degree of diffusion which has occurred at any given time.

59. ***Co-efficient of Diffusion.***—Experiment shows that the amount of a salt which at a given temperature passes, in unit time, through unit area of a horizontal surface, depends upon the nature of the salt and rate of change of concentration at that surface,—that is, the quantity of a salt that passes a given horizontal plane in unit time is κCA, where A is the area, C the rate of change of concentration, and κ a co-efficient that depends upon the nature of the substance. By rate of change of concentration is meant the difference in the quantities of salt in solution measured in grams per centimetre cube, at two horizontal planes one centimetre apart, supposing the concentration to diminish uniformly from one to the other. It is plain, that, if C and A in the above expression are unity, the quantity of salt passing in unit time is κ. κ, or the co-efficient of diffusion, is, therefore, the quantity of salt that passes in unit time through unit area of a horizontal plane when the difference of concentration is unity. Co-efficients of diffusion increase with the temperature, and are found not to be entirely independent of the degree of concentration.

As implied above, the units of mass and length employed in these measurements are respectively the gram and the centimetre; but, since in most cases the quantity of salt that diffuses in one second is extremely small, it is usual to employ the day as the unit time. The following examples will give an idea of the values of these co-efficients upon the basis of these units:—

Cane sugar at 15° temperature	0.3144
Common salt at 5° temperature	0.7650
Common salt at 15° temperature	0.9370
Common salt at 20° temperature	1.1330
Zinc sulphate at 24° temperature	0.2401
Albumen at 13° temperature	0.0630

60. ***Diffusion through Porous Bodies.*** — It was found by Graham that diffusion takes place through porous solids, such as unglazed earthenware, plaster, etc., almost as though the liquids were in direct contact, and that a very considerable difference of pressure could be established between the two faces of the porous body while the rate of diffusion remains nearly constant.

61. ***Diffusion through Membranes.*** — If the membrane through which diffusion occurs is of a type represented by animal or vegetable tissue, the resulting phenomena, though in some respects similar, are subject to quite different laws: thus, a certain class of non-crystallizable substances, represented by albumen, gelatine, etc., hence called *colloids*, pass through the membrane only to a very small extent; while those bodies capable of crystallization, called *crystalloids*, pass more freely. It is to be noted that the membrane is not a mere passive medium, as is the case with the porous substances already considered, but takes an active part in the process; and consequently one at least of the liquids frequently passes into the other more rapidly than would be the case if the surfaces of the liquids were directly in contact.

The explanation of these facts follows if we suppose that diffusion of a liquid through a continuous membrane can occur only when the liquid is capable of tem-

porarily uniting with the membrane, and forming a part of it. Diffusion occurs, therefore, by the union of the liquid with the membrane on one face, and the setting free of an equal portion on the other.

If the membrane separate two crystallizable substances, it often happens that both substances pass through, but at different rates. In accordance with the usage of Dutrochet, who carefully studied the phenomena, we may say there is *endosmose* of the liquid, which passes more rapidly to the other liquid, and *exosmose* of the latter to the former. The whole process is frequently called *osmosis*. If the membrane be stretched over the end of a tube, into which the more rapid current sets, and the tube be placed in a vertical position, the liquid will rise in the tube until a very considerable pressure is attained. Dutrochet called such an instrument an endosmometer.

Graham made use of a similar instrument, which he called an osmometer, by means of which he studied, not only the action of porous substances, such as are mentioned above, but also that of various organic tissues; and he was able to reach quantitative results of great value with respect to the substances employed. Pfeffer has more recently made an extended study of the phenomena of osmosis, especially in those aspects relating to physiological phenomena, in which he has shown that colloid membranes produced by purely chemical means — for example, ferrocyanide of copper — are even more efficient than the organic membranes employed by Graham.

62. **Dialysis.** — Upon the principles just set forth Graham has founded a method of separating crystalloids

from any colloid matters in which they may be contained, which is often of great importance in chemical processes. The apparatus employed by Graham consists of a hoop, over one side of which parchment paper is stretched so as to constitute a shallow basin. In this basin is placed the mixture under investigation, and the basin is then floated upon pure water contained in an outer vessel. If crystalloids are present, they will in due time pass through the membrane into the water, leaving the colloids behind. The process is often employed in toxicology for separating poisons from ingesta or other matters suspected of containing them. It is called *dialysis*, and the substances that pass through are said to dialyse.

DIFFUSION OF GASES.

63. ***Laws of Diffusion of Gases.***—Diffusion of gases obeys the same elementary laws as liquids. The rate of diffusion varies inversely as the pressure, directly as the square of the absolute temperature (see Book II.), and inversely as the square root of the density of the gas referred to its normal standard. A gas diffuses through porous solids according to the same laws. An apparatus by which this may be conveniently illustrated consists of a porous cell, whose open end is closed by a stopper, through which passes a long tube. This is placed in a vertical position, with the open end of the tube in a vessel of water. If, now, a bell-jar containing hydrogen be placed over the porous cell, hydrogen passes into the cell more rapidly than the air escapes from it: the pressure inside is increased, which is shown by the escape of bubbles from the end of the tube. If, now,

the jar be removed, diffusion outward occurs more rapidly than diffusion inward: the pressure within soon becomes less than the atmospheric pressure, as is shown by the rise of the water in the tube.

ELASTICITY.

64. ***Stress and Strain.*** — When a body is made the medium for the transmission of force, the application of Newton's third law shows that there is a stress in the medium. This stress is always accompanied in natural bodies by a corresponding change of form of the body, called a strain.

In some bodies equal stresses applied in any direction produce equal and similar strains. Such bodies are isotropic. In others the strain alters with the direction of the stress. These bodies are eolotropic.

65. ***Molecular Theory.*** — According to the molecular theory of matter, a body's form is permanent so long as the resultant of the stresses acting on it from without, with the attractions and repulsions existing between the individual molecules, reduces to zero. The repulsions and attractions vary by different laws as respects distance: consequently there is a certain form of the body for every external stress in which its molecules are in equilibrium. Any change of the stress in the body is accompanied by a re-adjustment of the molecules, which is continued until equilibrium is again established.

If the stress tends only to increase or diminish the distance between the molecules, it is called a tension or a pressure respectively; if it tends to slide one line or sheet of molecules past another tangentially, it is

called a shear or a shearing-stress. All stresses can be resolved into these two forms. The corresponding changes of shape are called dilatations, compressions, and shearing-strains.

66. ***Modulus of Elasticity.*** — If for a given amount of stress between certain limits a body is deformed by a definite amount, which is constant so long as the stress remains constant, and if, when the stress is removed, the body regains its original condition, it is said to be perfectly elastic. Any body only partially fulfilling these conditions is said to be imperfectly elastic.

The definition of elasticity in its physical sense, as a property of bodies, has been already given. As a measurable quantity, it may be defined as the rate of change, in a unit of the body, of the stress with respect to the strain. Thus, for example, the voluminal or bulk elasticity of a fluid is the ratio of any small change of pressure to the corresponding change of unit volume. The tractional elasticity of a wire under tension is the ratio of any small change in the stretching-weight to the corresponding change in unit length. This ratio expressed numerically is called the modulus, and its reciprocal the co-efficient, of elasticity.

67. ***Modulus of Voluminal Elasticity of Gases.*** — All gases within certain limits of temperature and pressure obey very closely the following law, discovered by Boyle in 1662, and afterwards fully proved and discussed by Mariotte: —

Boyle's Law. — The temperature remaining the same, the volume of any gas is inversely as the pressure upon it.

Thus, if P and P_1 represent different pressures, V and V_1 the corresponding volumes of any gas, then

$$P : P_1 = \frac{1}{V} : \frac{1}{V_1};$$

whence

$$PV = P_1 V_1. \qquad (27)$$

Now, $P_1 V_1$ is a constant which may be determined by choosing any pressure P_1 and the corresponding volume V_1 as standards: hence we may say, that, at any given temperature, the product PV is a constant. The limitations of this law will be noticed later.

If we draw the curve marked out by a point whose ordinate and abscissa are so related that xy equals a constant, we obtain a rectangular hyperbola referred to its asymptotes. Let x represent the volume and y the pressure of a quantity of gas. Then this curve shows the relation of pressure and volume in all their combinations.

Draw the lines as in Fig. 38, letting AC, JD, represent volumes differing only by a small amount.

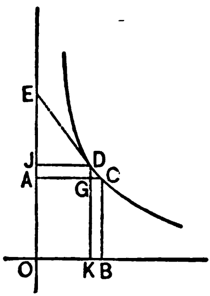

Fig. 38.

It can be shown that AE is numerically equal to the modulus of elasticity, for $\frac{CG}{AC}$ is the voluminal compression per unit volume for the increment of pressure GD: hence, by definition, $\frac{GD}{\frac{CG}{AC}}$ is the modulus of elasticity. But

$$AE : GD = AC : CG,$$

from similarity of triangles: hence

$$AE = \frac{GD}{\frac{CG}{AC}} = \text{the modulus of elasticity.}$$

Further, since the rectangles JG and CK are equal,

$$DG \cdot GA = CG \cdot GK;$$

whence

$$CG : DG = GA : GK.$$

By similar triangles,

$$CG : DG = CA : AE;$$

whence

$$GA : GK = CA : AE.$$

Now, if the increment of pressure be made very small, so that D and C ultimately coincide, the line CE becomes a tangent, and GA, GK, are respectively equal to CA, CB. CB therefore equals AE from the last proportion: hence, in the case of a gas obeying Boyle's law, the modulus of elasticity is numerically equal to the pressure.

The discussion of the experimental facts in connection with the elasticity of gases, and the explanation of the apparatus founded upon it will be resumed in a future chapter.

68. ***Modulus of Voluminal Elasticity of Liquids.***—When liquids are subjected to voluminal compression, it is found that their modulus of elasticity is much greater than that of gases. For at least a limited range of pressures the modulus of elasticity of any one liquid is constant, the change in volume being proportional to the change in the pressure. The modulus differs for different liquids.

The instrument used to determine the modulus of elasticity of liquids is called a piezometer. The first form in which the instrument was devised by Oersted, while not the best for accurate determinations, may yet serve as a type.

The liquid to be compressed is contained in a thin glass flask, the neck of which is a tube with a capillary bore. The flask is immersed in water contained in a strong glass vessel fitted with a water-tight metal cap, through which moves a piston. By the piston, pressure may be applied to the water, and through it to the flask and to the liquid contained in it.

The end of the neck of the small flask is inserted downwards under the surface of a quantity of mercury which lies at the bottom of the stout vessel. The pressure is registered by means of a compressed-air manometer (see § 86) also inserted in the vessel. When the apparatus is arranged, and the piston depressed, a rise of the mercury in the neck of the flask occurs, which indicates that the water has been compressed.

An error may arise in the use of this form of apparatus from the change in the capacity of the flask, due to the pressure. Oersted assumed, since the pressure on the interior and exterior walls was the same, that no change would occur. Lamé, however, showed that such a change would occur, and gave the formulas by which it might be calculated. By thus introducing the proper corrections, Oersted's piezometer may be used with success.

A different form of the instrument, employed by Regnault, is, however, to be preferred. In it, by an arrangement of stop-cocks, it is possible to apply the

pressure upon either the interior or exterior wall of the flask separately, or upon both together, and in this way to experimentally determine the correction to be applied for the change in the capacity of the flask.

It is to be noted that the modulus of elasticity for liquids is so large, that, within the ordinary range of pressures, they may be regarded as incompressible. Thus, for example, the alteration of volume for sea-water by the addition of one atmosphere's pressure is 0.000044. The change in volume, then, at a depth in the ocean of one kilometre, where the pressure is 93 atmospheres nearly, is 0.0041, or about $\frac{1}{250}$ of the whole volume.

69. ***Modulus of Voluminal Elasticity of Solids.***—The modulus of voluminal elasticity of solids is believed to be generally greater than that of liquids, though no reliable experimental results have yet been obtained.

The modulus, as is the case with liquids, differs for different bodies.

70. ***Shears.***—A strain in which parallel planes or sheets of molecules are moved tangentially over one another, each plane being displaced by an amount proportional to its distance from one of the planes assumed as fixed, is called a shear.

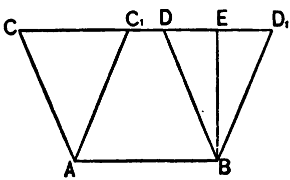

Fig. 39.

To illustrate this definition, let us consider a parallelopiped, whose cross-section made at right angles to its sides is a rhombus, and let *AB DC* in the diagram (Fig. 39) represent that cross-section.

If the rhombus *ABDC* is deformed so as to become

ABD_1C_1, that deformation is a simple shear. It is plain that a simple shear is equivalent to an extension in lines parallel to AD, and a contraction in those at right angles to AD. The directions AD and CB are called the principal axes of the shear. The amount of the shear is the displacement of the planes per unit of distance from the fixed plane; that is, $\frac{DD_1}{EB}$ is the amount of the shear.

The stresses that give rise to a simple shear can plainly be conceived of as consisting of two equal couples, the forces comprising which act tangentially upon parallel planes which are moved over one another, and make equal angles with the axes of the shear. The forces making up these couples may be compounded two and two, a and b, a_1 and b_1 (Fig. 40), making up a tension normal to the diminished axis; a_1 and b, a and b_1, making up a pressure normal to the increased axis. These stresses are measured per unit of area of the undeformed sides or sections of the solid.

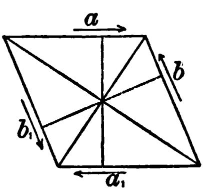

Fig. 40.

The resistance offered by a body to a shearing-stress is called its rigidity, and the ratio of a very small change in the stress to the corresponding increment in the amount of the shear is called the modulus of rigidity.

71. ***Elasticity of Tension.***—The first experimental determinations of the relations between the elongation of a solid and the tension acting on it were made by Hooke in 1678. Experimenting with wires of different materials, he found that for small tensions the

elongation is proportional to the stress. It was afterwards found that this law is true for small compressions.

The ratio of the stress to the elongation of unit length of a wire of unit section is the modulus of tractional elasticity. For different wires it is found that the elongation is proportional to the length of the wires, and inversely to their section. The formula embodying these facts is

$$e = \frac{Sl}{\mu s}, \qquad (28)$$

where e is the elongation, l the length, s the section of the wire, S the stress, and μ the modulus of tractional elasticity.

A method of expressing the modulus of elasticity, due to Thomas Young, is sometimes valuable. "We may express the elasticity of any substance by the weight of a certain column of the same substance, which may be denominated the modulus of its elasticity, and of which the weight is such that any addition to it would increase it in the same proportion as the weight added would shorten, by its pressure, a portion of the substance of equal diameter." For example, considering a cubic litre of air at 0° C. and 760 millimetres of mercury pressure, and calling its weight unity, we find, from the fact that the weight of one litre of mercury is 10517 times that of a litre of air, that the pressure of the column of air upon a square decimetre is 79929 units. If we conceive the air as of equal density throughout, this pressure is equivalent to the weight of a column of air one square decimetre in section and 7992.9 metres high. The weight of this

column is the modulus of elasticity for air; for we know, by Boyle's law, that if the column be altered in length, and its weight therefore correspondingly altered, the volume of the cubic litre of air under consideration will also alter inversely. The height of such a column of air as we have assumed is called the height of the homogeneous atmosphere.

72. ***Elasticity of Torsion.*** — When a cylindrical wire clamped at one end is subjected at the other to the action of a couple whose axis is the axis of the cylinder, it is found that the amount of torsion (measured by the angle of displacement of the arm of the couple) is proportional to the moment of the couple, to the length of the wire, and inversely to the fourth power of the radius. It also depends on the modulus of rigidity. The formulated statement of these facts is

$$\phi = \frac{Cl}{nr^4}, \tag{29}$$

where ϕ is the angle of torsion, l the length, r the radius of the wire, C the moment of couple, and n the modulus of rigidity. No general formula can be found for wires whose sections are of variable form.

The laws of torsion in wires were first investigated by Coulomb, who applied them in the construction of an apparatus of great value for the measurement of small forces.

The apparatus consists essentially of a small cylindrical wire, suspended firmly from the centre of a disk, upon which is cut a graduated circle. By the rotation of this disk any required amount of torsion may be given to the wire. On the other extremity of the wire

is fixed, horizontally, a bar, to the ends of which the forces constituting the couple are applied. Arrangements are also made by which the angular deviation of this bar from the point of equilibrium may be determined. When forces are applied to the bar, it is brought back to its former point of equilibrium by rotation of the upper disk. Let Θ represent the moment of torsion (that is, the couple which, acting on an arm of unit length, will give the wire an amount of torsion equal to a radian), C the moment of couple acting on the bar, τ the amount of torsion measured in radians; then

$$C = \Theta\tau.$$

We may find the value of Θ in absolute measure by a method of oscillations analogous to that used to determine g with the pendulum.

A body whose moment of inertia can be determined by calculation is substituted for the bar, and the time t of one of its oscillations about the position of equilibrium observed. We then obtain Θ from the formula

$$t = \pi\sqrt{\frac{I}{\Theta}};$$

for the amount of torsion is proportional to the moment of couple, — a condition giving a simple harmonic motion to the oscillating body. The velocity of any point of distance r from the axis of rotation, therefore, is $\frac{2\pi a}{t}$ at the point of equilibrium, and its angular velocity is $\frac{2\pi\tau}{t}$, for $a = r\tau$. The kinetic energy of a body rotating about a centre is $\frac{1}{2}I\omega^2$; and the kinetic

energy of the body considered at the point of equilibrium is, therefore,

$$\tfrac{1}{2}I \cdot \frac{4\pi^2\tau^2}{t^2}.$$

The potential energy due to the torsion of the wire is $\frac{1}{2}\Theta\tau^2$, where $\frac{1}{2}\Theta\tau$ is the average moment of couple, and τ the distance through which this couple acts. These expressions are necessarily equal: hence

$$\frac{I 4\pi^2\tau^2}{2t^2} = \tfrac{1}{2}\Theta\tau^2,$$

or

$$\Theta = \frac{4\pi^2 I}{t^2}.$$

We may use a single instead of a double oscillation, when we may write the formula

$$\Theta = \frac{\pi^2 I}{t^2}. \qquad (30)$$

This apparatus was used by Coulomb in his investigations of the law of electrical and magnetic actions. It was afterwards employed by Cavendish, as has been already noticed, to determine the constant of gravitation.

73. ***Elasticity of Flexure.*** — If a rectangular bar be clamped by one end, and acted on at the other by a force normal to one of its sides, it will be bent or flexed. The amount of flexure — that is, the amount of displacement of the extremity of the bar from its original position — is found to be proportional to the force, to the cube of the length of the bar, and inversely to its breadth, to the cube of its thickness, and

to the modulus of tractional elasticity. The formula therefore becomes

$$f = \frac{Fl^3}{\mu b d^3}. \qquad (31)$$

74. ***Limits of Elasticity.***—The theoretical deductions and empirical formulas which we have hitherto been considering are strictly applicable only to perfectly elastic bodies. It is found that the voluminal elasticity of fluids is perfect, and that within certain limits of deformation, varying for different bodies, we may consider the elasticity of solids to be practically perfect for every kind of strain. If the strain be carried beyond the limit of perfect elasticity, the body is permanently deformed. This permanent deformation is called set.

Upon these facts we may base a distinction between solids and fluids: a solid always requires the stress acting on it to exceed a certain limit before any permanent set occurs, and it makes no difference how long the stress acts provided it lies within the limits. A fluid, on the contrary, may be deformed by the slightest shearing-stress, provided time enough be allowed for the movement to take place. The fundamental difference lies in the fact that fluids offer no resistance to shearing-stress other than that due to internal friction or viscosity.

A solid, if it be deformed by a slight stress, is soft; if only by a great stress, is hard or rigid. A fluid, if deformed quickly by any stress, is mobile; if slowly, is viscous.

It must not be understood, however, that the behavior of elastic solids under stress is entirely independent of

time. If, for example, a steel wire be stretched by a weight which is nearly but not quite sufficient to produce an immediate set, it is found, that, after some time has elapsed, the wire acquires a permanent set. If, on the other hand, a weight is put upon the wire somewhat less than is required to break it, by allowing intervals of time to elapse between the successive additions of small weights, the total weight supported by the wire may be raised considerably above the breaking-weight. Again: if the weight stretching the wire be removed, the return to its original form is not immediate, but gradual. If a wire carrying a weight is twisted, and the weight set oscillating by the torsion of the wire, it is found that the oscillations die away faster than can be explained by any imperfections in the elasticity of the wire.

These and similar phenomena are manifestly dependent upon peculiarities of molecular arrangement and motion. The last two are exhibitions of the "viscosity of solids." The molecules of solids, just as those of liquids, move among themselves, but with a certain amount of frictional resistance. This resistance causes the external work done by the body to be diminished, and the internal work done among the molecules becomes transformed into heat.

CHAPTER IV.

MECHANICS OF FLUIDS.

75. ***Pascal's Law.***—A perfect fluid may be defined as a body which offers no resistance to shearing-stress. No actual fluids are perfect. Even those which approximate that condition most nearly, offer resistance to shearing-stress, due to their viscosity. With most, however, a very short time only is needed for this resistance to vanish; and all mobile fluids at rest can be dealt with as if they were perfect in determining the conditions of equilibrium. If they are in motion, their viscosity becomes a more important factor.

As a consequence of this definition of a perfect fluid, follows a most important deduction. It was first made by Pascal, and is known as Pascal's law. "In a fluid in equilibrium, not acted on by any outside forces except the pressure of the containing vessel, the pressure at every point and in every direction is the same." By pressure at a point is meant the ratio of the pressure on an area to that area taken with its centre of inertia at the point, and normal to the direction of the pressure, when the area is diminished indefinitely.

The truth of Pascal's law appears, if, in a fluid fulfilling the conditions indicated, we consider a cube of the fluid to become solidified. Then if the law as just

stated were not true, there would be an unbalanced force in some direction, and the cube would move, which is contrary to the statement that the fluid is in equilibrium. If a vessel filled with a fluid be fitted with a number of pistons of equal area A, and a force Ap be applied to one of them, acting inwards, a pressure Ap will act outwards upon the face of each of the pistons. These pressures may be balanced by equal forces applied to each piston. If $n + 1$ be the number of the pistons, the outward pressure on n of them, caused by the force applied to one, is npA.

The fluid will be in equilibrium when a force p is acting on unit area of each piston. It is plain that the same reasoning will hold if the area of one of the pistons is A and of another is nA. A pressure Ap on the one will balance a pressure of nAp on the other. This principle governs the action of the hydrostatic press.

76. ***Relations of Fluid Pressures due to Outside Forces.*** — If forces, such as gravitation, act on the mass of a fluid from without, Pascal's law no longer holds true. For consider the unit cube of solidified fluid to be acted on by gravity; then the pressure on the upper face must be less than that on the lower face by the weight of the cube, in order that the fluid may still be in equilibrium. As the cube may be made as small as we please, it appears that in the limit the pressure on the two faces only differs by an infinitesimal; that is, the pressure in a fluid acted on by outside forces is the same at one point for all directions, but varies continuously for different points.

The surface of a fluid of uniform density acted on by gravity, if at rest, is everywhere perpendicular to the

lines of force; for, if this were not so, the force at a point on the surface could be resolved into two components, one perpendicular and the other tangent to the surface. But from the nature of a fluid the tangential force would set up a motion of the fluid, which is contrary to the statement that the fluid is at rest. If a surface be drawn through the points in the field at which the pressure is the same, that surface will be perpendicular to the lines of force. For consider a filament of solidified fluid lying in the surface: its two ends suffer equal and opposite pressures; hence, since by hypothesis the fluid is in equilibrium, the force acting upon it, due to gravity, can have no component in the direction of its length, and is perpendicular to the surface in which it lies.

Surfaces of equal pressures are equipotential surfaces. In small masses of fluid, in which the lines of force due to gravity are parallel, these surfaces are horizontal planes. In larger masses, such as the oceans, they are curved to correspond to the divergence of the lines of force from the centre of the earth.

In a liquid, which is practically incompressible, the pressure at a point is proportional to its depth below the surface of the liquid; for, if we consider two rectangular prisms of solidified liquid whose bases are equal, and the length of the one n times that of the other, and whose bases are coincident with the surface of the liquid, it is clear that equilibrium can exist only when the upward pressure on the base of the one is n times that on the other. For the pressure on the base is balanced by the force due to gravity, which is proportional to the mass of the prism. But the pressure

on the base of the one, being *n* times that on the other, is proportional to the depth of the base; for the altitude of the one is *n* times that of the other.

From the foregoing principles, it is evident that a liquid contained in two communicating vessels of any shape whatever will stand at the same level in both. If one, however, be filled with a liquid of different density from that in the other, equilibrium will be established when the depths are inversely as the densities of the liquids.

77. ***The Barometer.***—The instrument best adapted to illustrate these principles, and also of great importance in many physical investigations, is the barometer. It was invented by Torricelli, a pupil of Galileo. The fact that water can be raised in a tube in which a complete or partial vacuum has been made was known to the ancients, and was explained by them, and by the Schoolmen after them, by the maxim that "Nature abhors a vacuum." Pumps must have been common; for the force-pump, a far more complicated instrument, was invented by Ctesibius of Alexandria, who lived during the second century B.C. It was not until the time of Galileo, however, that the first recorded observations were made that the column of water in a pump could rise only to a height of 10.5 metres. Galileo failed to give the true explanation of this fact. He had, however, taught that the air has weight; and his pupil Torricelli, using that principle, was more successful.

He showed, that if a glass tube sealed at one end, over 760 millimetres long, were filled with mercury, the open end stopped with the finger, the tube inverted, and the unsealed end plunged beneath a surface of mercury

in a basin, on withdrawing the finger the mercury in the tube sank till its top surface was about 760 millimetres above the surface of the mercury in the basin. The specific gravity of the mercury being 13.59, the weight of the mercury column and that of the water column in the pump agreed so nearly as to show that the maintenance of the columns in both cases was due to a common cause, — the pressure of the atmosphere. This conclusion was subsequently verified and established by Pascal, who requested a friend to observe the height of the mercury column at the bottom and at the top of a mountain. On making the observation, the height of the column at the top was found to be less than at the bottom. Pascal himself afterwards observed a slight though distinct diminution in the height of the column on ascending the tower of St. Jacques de la Boucherie in Paris.

The form of barometer first made by Torricelli is still often used, especially when the instrument is stationary, and is intended to be one of precision. In the finest instruments of this class a tube is used which is three or four centimetres in diameter, so as to avoid any correction for capillarity. A screw of known length, pointed at both ends, is arranged so as to move vertically above the surface of the mercury in the cistern. When an observation is to be made, the screw is moved until its lower point just touches the surface. The distance between its upper point and the top of the column is measured by means of a cathetometer; and this distance added to the length of the screw gives the height of the column.

Other forms of the instrument are used, most of

which are arranged with reference to convenient transportability. Various contrivances are added by means of which the movement of the column is made to move an index, and thus record the pressure on a graduated scale. All these forms are only modifications of Torricelli's original instrument.

The pressure indicated by the barometer is usually stated in terms of the height of the column. Mercury being practically incompressible, this height is manifestly proportional to the pressure at any point in the surface of the mercury in the cistern. The actual pressure on any given area in that surface can be calculated if we know the value of g at the place and the specific gravity of mercury, as well as the height of the column. The standard barometric pressure, represented by 760 millimetres of mercury, is a pressure of 1.033 kilograms on every square centimetre. It is often called a pressure of one atmosphere; and larger pressures are measured by "atmospheres."

In the preparation of an accurate barometer, it is necessary that all air be removed from the mercury: otherwise it will collect in the upper part of the tube, by its pressure lower the top of the column, and make the barometer read too low. The air is removed by partially filling the tube with mercury, which is then boiled in the tube, gradually adding small quantities of mercury, and boiling after each addition, until the tube is filled. The boiling must not be carried too far; for there is danger, in this process, of expelling the air so completely that the mercury will adhere to the sides of the tube, and will not move freely. For rough work the tube may be filled with cold mercury, and the air

removed by gently tapping the tube, so inclining it that the small bubbles of air which form can coalesce, and finally be set free at the surface of the mercury.

78. ***Archimedes' Principle.*** — If a solid be immersed in a fluid, it loses in weight an amount equal to the weight of the fluid displaced. This law is known, from its discoverer, as Archimedes' Principle.

The law appears evidently true if we consider the space occupied by the solid as filled with the fluid. The fluid in this space will then be in equilibrium, and the upward pressure on it must exceed the downward pressure by an amount equal to its weight. The resultant of the pressure acts through the centre of gravity of the assumed portion of fluid, otherwise equilibrium would not exist. If, now, the solid occupy the space, the difference between the upward and the downward pressures on it must still be the same as before, — namely, the weight of the fluid displaced by the solid; that is, the solid loses in apparent weight an amount equal to the weight of the displaced fluid.

79. ***Floating Bodies.*** — When the solid floats on the fluid, the weight of the solid is balanced by the upward pressure. In order that the solid shall be in equilibrium, these forces must act in the same line. The resultant of the pressure which lies in the vertical line passing through the centre of gravity of the displaced fluid must pass through the centre of gravity of the solid. Draw the line in the solid joining these two centres, and call it the axis of the solid. The equilibrium is stable when, for any infinitesimal inclination of the axis from the vertical, the vertical line of upward pressure cuts the axis in a point above the centre of

gravity of the solid. This point is called the meta-centre.

80. ***Specific Gravity.***—Archimedes' principle is used to determine the specific gravity of bodies. The specific gravity of a body is defined as the ratio of its weight to the weight of an equal volume of pure water at a standard temperature.

The specific gravity of a solid that is not acted on by water may be determined by means of the hydrostatic balance. The body under examination, if it will sink in water, is suspended from one scale-pan of a balance by a fine thread, and is weighed. It is then immersed in water, and is weighed again. The difference between the weights in air and in water is the weight of the displaced water, and the ratio of the weight of the body to the weight of the displaced water is the specific gravity of the body. If the body will not sink in water, it may be joined to some known weight, sufficient to sink it, of a body whose specific gravity is already determined. The specific gravity of the combination being obtained as before, that of the body in question is easily calculated.

Or a sinker of unknown weight and specific gravity may be suspended from the balance, and counterpoised in water. Now, if the body whose specific gravity is sought be attached to the sinker, it will be found that the equilibrium is destroyed. To restore it, weights must be added to the same side. These, being added to the weight of the body, represent the weight of the water displaced.

The specific gravity of a liquid is obtained by first weighing in air a mass of some solid, such as platinum

or glass, that is not acted on chemically by the liquid, and then weighing the mass successively in the liquid to be tested and in water. The ratio of the differences between these weights and the weight of the body in the air is the specific gravity of the liquid.

The specific gravity of a liquid may also be found by means of the specific-gravity bottle. This is a bottle fitted with a ground-glass stopper. The weight of the water which completely fills it is determined once for all. When the specific gravity of any liquid is desired, the bottle is filled with the liquid, and the weight of the liquid determined. The ratio of this weight to the weight of an equal volume of water is the specific gravity of the liquid.

The same bottle may be used to determine the specific gravity of any solid which cannot be obtained in continuous masses, but is friable or granular. A weighed amount of the solid is introduced into the bottle, which is then filled with water, and the weight of the joint contents of the bottle determined. The difference between the last weight and the sum of the weights of the solid and of the water filling the bottle is the weight of the water displaced by the solid. The ratio of the weight of the solid to the weight thus obtained is the specific gravity of the solid.

The specific gravity of a liquid may also be obtained by means of hydrometers. These are of two kinds, — the hydrometers of constant weight and those of constant volume. The first consists usually of a glass bulb surmounted by a cylindrical stem. The bulb is weighted with mercury or shot, so as to sink in pure water to some definite point on the stem. This point is taken

as the zero; and, by successive trials with different liquids of known specific gravity, points are found on the stem empirically to which the hydrometer sinks in these liquids. With these as a basis, the divisions of the scale are determined and cut on the stem.

The hydrometer of constant volume consists of a bulb weighted so as to stand upright in the liquid, bearing on the top of a narrow stem a small pan, in which weights may be placed. The weight of the hydrometer being known, it is immersed in water; and, by the addition of weights in the pan, a fixed point on the stem is brought to coincide with the surface of the water. The instrument is then transferred to the liquid to be tested, and the weights in the pan changed until the fixed point again comes to the surface of the liquid. The sum of the weight of the hydrometer and the weights added in each case gives the weight of equal volumes of water and of the liquid, from which the specific gravity sought is easily obtained.

The specific gravity of gases is often referred to air or to hydrogen instead of water. It is best determined by filling a large glass flask, whose weight is known, with the gas whose specific gravity is to be obtained, carefully dried, and weighing it, noting the temperature and the pressure of the gas in the flask. The weight of the gas at the standard temperature and pressure is then calculated, and the ratio of this weight to the weight of the same volume of the standard gas is the specific gravity desired. The weight of the flask used in this experiment must be very exactly determined. The presence of the air vitiates all weighings performed in it, by diminishing the real weight of the body to be

weighed and of the weights employed, by an amount proportional to their volumes. The consequent error is avoided either by performing the weighings in a vacuum produced by the air-pump, or by correcting the apparent weight in air to the real weight. Knowing the specific gravity of the weights and of the body to be weighed, and the specific gravity of air, this can easily be done.

81. ***Motions of Fluids.***—Thus far we have considered fluids in equilibrium. If the parts of the fluid are moving relatively to each other or to its bounding-surface, the circumstances of the motion can be determined only by making limitations which are not actually found in nature. There thus arise certain definitions to which we assume that the fluid under consideration conforms.

The motion of a fluid is said to be uniform when each element of it has the same velocity at all points of its path. The motion is steady when at any one point the velocity and direction of motion of the elements successively arriving at that point remain the same for each element. If either the velocity or direction of motion changes for successive elements, the motion is said to be varying. The motion is further said to be rotational or irrotational according as the elements of the fluid have or have not an angular velocity about their axes.

In all discussions of the motions of fluids a condition is supposed to hold, called the condition of continuity. It expresses the fact, that, in any portion of space selected in the fluid, the change of density in that space depends solely on the difference between the amounts of fluid flowing into and out of that space. In an in-

compressible fluid or liquid, if the influx is reckoned plus and the outflow minus, we have, letting Q represent the amount of the liquid passing through the boundary in any one direction, $\Sigma Q = 0$. The results obtained in the discussion of fluid motions must all be interpreted consistently with this condition. If the motion is such that the fluid breaks up into discontinuous parts, any results obtained by hydrodynamical considerations no longer hold true.

If we consider any stream of incompressible fluid, of which the cross-sections at two points where the velocities of the elements are v_1 and v_2 have respectively the areas A_1 and A_2, we can deduce at once from the condition of continuity

$$A_1 v_1 = A_2 v_2. \tag{32}$$

82. ***Velocity of Efflux.***—We shall now apply this principle to discover the velocity of efflux of a liquid from an orifice in the walls of a vessel, the motion of the fluid being irrotational.

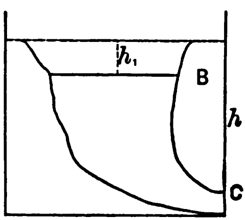

Fig. 41.

Consider any small portion of the liquid, bounded by stream lines, which we may call a filament. The velocity of the filament at B (Fig. 41) is v_1, and at C is v, the areas of the cross-sections of the elements at the same points being A_1 and A. We have then, as above, $A_1 v_1 = Av$. We assume that the flow has been established for a time sufficiently long for the motion to become steady. The energy of the mass contained in the filament between B and C is, therefore, constant. Let

V_1 be the potential at B due to gravity, V the potential at C, and d the density of the liquid. The mass that enters at B in a unit of time is

$$dA_1v_1 = dAv,$$

the mass that goes out at C. The energy entering at B is

$$dA_1v_1(\tfrac{1}{2}v_1^2 + V_1),$$

the energy passing out at C is

$$dAv(\tfrac{1}{2}v^2 + V).$$

If the pressures at B and C on unit areas are expressed by p_1 and p, the work done at B on the entering mass by the pressure p_1 is $p_1A_1v_1$, and at C on the outgoing mass is pAv. The energy within the filament remaining constant, the incoming must equal the outgoing energy; therefore

$$pAv + dAv(\tfrac{1}{2}v^2 + V) = p_1A_1v_1 + dA_1v_1(\tfrac{1}{2}v_1^2 + V_1),$$

whence

$$\frac{p}{d} + \tfrac{1}{2}v^2 + V = \frac{p_1}{d} + \tfrac{1}{2}v_1^2 + V_1.$$

We may write this equation

$$\tfrac{1}{2}(v^2 - v_1^2) = (V_1 - V) + \frac{p_1 - p}{d}, \qquad (33)$$

or, again, since $Av = A_1v_1$,

$$\tfrac{1}{2}v^2\left(1 - \frac{A^2}{A_1^2}\right) = (V_1 - V) + \frac{p_1 - p}{d}. \qquad (34)$$

Applying equation (34) to the case of a liquid flowing freely into air from an orifice at C, we have

$$(V_1 - V) = g(h - h_1),$$

where h is the height of the surface above C, and h_1 is that of the surface above B;

$$p_1 = p_a + dgh_1,$$

where p_a is the atmospheric pressure. At the orifice $p = p_a$. We have then

$$\tfrac{1}{2}v^2\left(1 - \frac{A^2}{A_1^2}\right) = g(h - h_1) + gh_1 = gh,$$

whence

$$v^2 = \frac{2gh}{1 - \frac{A^2}{A_1^2}}.$$

If, now, A is very small as compared with A_1, the velocity at C becomes

$$v = \sqrt{2gh}; \qquad (35)$$

that is, the velocity of efflux of a small stream issuing from an orifice in the wall of a vessel is independent of the density of the liquid, and is equal to the velocity which a body would acquire in falling freely through a distance equal to that between the surface of the liquid and the orifice.

This theorem was first given by Torricelli on experimental considerations, and is known by his name.

We may apply the general equation to the case of the efflux of a liquid through a siphon. A siphon is a bent tube, one limb of which is longer than the other. It is filled with liquid, stopped with the finger, inverted, and the end of the shorter limb immersed in some of the liquid contained in a vessel. On removing the finger the liquid runs from the longer limb.

In this case, as before, $\frac{A^2}{A_1^2} = 0$, $v_1 = 0$, p and p_1

both $= p_a$, and $(V_1 - V) = gl$, where l is the distance between the surface level and the orifice of the longer limb. The velocity becomes $v = \sqrt{2gl}$. The siphon, therefore, discharges more rapidly the greater the distance between the surface level and the orifice. It is manifest that the height of the bend in the tube cannot be greater than that at which atmospheric pressure would support the liquid.

The ordinary lifting-pump may also be discussed by the same equation. This pump consists essentially of a tube, fitted near the bottom with a partition, in which is a valve opening upwards. In the tube slides a tightly fitting piston, in which is a valve, also opening upwards. The piston is first driven down to the partition in the tube, the enclosed air escaping through the valve in the piston. When the piston is raised, the liquid in which the lower end of the tube is immersed passes through the valve in the partition, and fills the tube. When the piston is again lowered, the space above it is filled with the liquid, which is lifted out of the tube at the next up-stroke.

To determine the velocity of the liquid following the piston, we notice that in this case $p_1 = p_a$ and $p = 0$ if the piston moves upward very rapidly, $(V_1 - V) = -gh$, where h is the height of the top of the liquid column above the free surface in the reservoir, and $\frac{A^2}{A_1^2}$ again $= 0$. We then have

$$\tfrac{1}{2}v^2 = \frac{p_a}{d} - gh.$$

The velocity when $h = 0$ is

$$v = \sqrt{\frac{2p_a}{d}},$$

and at other points it can be found by suitable substitutions. When h is such that $dgh = p_a$, $v = 0$, which expresses the condition of equilibrium.

Torricelli's theorem is shown to be approximately true by allowing liquids to run from an orifice in the side of a vessel, and measuring the path of the stream. If the theorem be true, this ought to be a parabola, of which the intersection of the plane of the stream and of the surface of the liquid is the directrix; for each portion of the liquid, after it has passed the orifice, will behave as a solid body, and move in a parabolic path. The equation of this path is found, as in § 37, to be

$$\frac{2v^2}{g}x = y^2.$$

Now, by Torricelli's theorem, we may substitute for v^2 its value $2gh$, $\therefore y^2 = 4hx$; whence, since the initial movement of the stream is supposed to be horizontal, the perpendicular line through the orifice being the axis of the parabola, and the orifice being the origin, h is the distance from the orifice to the directrix. Experiments of this kind have been frequently tried, and the results found to approximate more nearly to the theoretical as various causes of error were removed.

When, however, we attempt to calculate the amount of liquid discharged in a given time, there is found to be a wider discrepancy between the results of calculation and the observed facts. Newton first noticed that the diameter of the jet at a short distance from the orifice is less than that of the orifice. He showed this to be a consequence of the freedom of motion among the particles in the vessel. The particles flow from all

directions toward the orifice, those moving from the sides necessarily issuing in streams inclined towards the axis of the jet. Newton showed, that by taking the diameter of the narrow part of the jet, which is called the *vena contracta*, as the diameter of the orifice, the calculated amount of liquid escaping agreed far more closely with theory.

If the orifice is fitted with a short cylindrical tube, the interference of the different particles of the liquid is in some degree lessened, and the quantity discharged increases nearly to that required by theory.

83. ***Diminution of Pressure.***—The Sprengel air-pump, an important piece of apparatus, to be described hereafter, depends for its operation on the diminution of pressure at points along the line of a flowing column of liquid. Let us consider a large reservoir filled with liquid, which runs from it by a vertical tube entering the bottom of the reservoir. Transposing equation (34) to find the value of p, we have

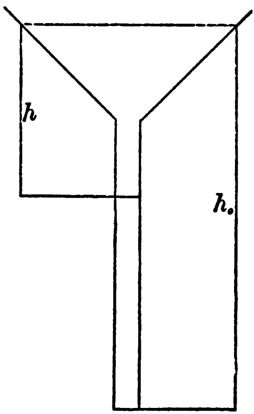

Fig. 42.

$$p = p_1 + (V_1 - V)d - \tfrac{1}{2}dv^2\left(1 - \frac{A^2}{A_1^2}\right).$$

The ratio $\dfrac{A^2}{A_1^2}$ is again 0. If h (Fig. 42) represent the height of the upper surface above the point in the tube at which we desire to find the pressure, then $(V_1 - V) = gh$. We then have $p = p_1 + dgh - \frac{1}{2}dv^2$. If the tube is always filled with the liquid, $Av = A_0v_0$, where A and A_0 are the areas of the cross-sections of the tube at the point

we are considering and at the bottom of the tube, and v and v_0 are the corresponding velocities. Further, $v_0^2 = 2gh_0$ if h_0 is the distance from the upper surface to the bottom of the tube. We obtain, by subtitution,

$$p = p_1 + dg\left(h - \frac{A_0^2}{A^2} h_0\right). \tag{36}$$

If $h = \frac{A_0^2}{A^2} h_0$, $p = p_1 = p_a$; and if an opening were made in the wall of the tube, the moving liquid and the air would be in equilibrium. If $h < \frac{A_0^2}{A^2} h_0$, the pressure p would be less than p_a, and air would flow into the tube. Since this inequality exists when $A_0 = A$, it follows, that, if a liquid flow from a reservoir down a cylindrical tube, the pressure at any point in the wall of the tube is less than the atmospheric pressure by an amount equal to the pressure of a column of the liquid the height of which is equal to the distance between the point considered and the bottom of the tube.

84. ***Vortices.*** — A most interesting series of results have been obtained, by Helmholtz, Thomson, and others, from the discussion of the rotational motions of fluids. Though the proofs are of such a nature that they cannot be presented here, the results are so important that they will be briefly stated.

A vortex line is defined as the line which coincides at every point with the instantaneous axis of rotation of the fluid element at that point. A vortex filament is any portion of the fluid bounded by vortex lines.

A vortex is a vortex filament which has "contiguous to it over its whole boundary irrotationally moving fluid."

The theorems relating to this form of motion as first proved by Helmholtz, in 1868, show that,—

(1) A vortex in a perfect fluid always contains the same fluid elements, no matter what its motion through the surrounding fluid may be.

(2) The "strength" of a vortex, which is the product of its angular velocity by its cross-section, is constant: therefore the vortex in an infinite fluid must always be a closed curve, which, however, may be knotted and twisted in any way whatever.

(3) In a finite fluid the vortex may be open, its two ends terminating in the surface of the fluid.

(4) The irrotationally moving fluid around a vortex has a motion due to its presence, and transmits the influence of the motion of one vortex to another.

(5) If the vortices considered are infinitely long and rectilinear, any one of them, if alone in the fluid, will remain fixed in position.

(6) If two such vortices be present parallel to one another, they revolve about their common centre of gravity.

(7) If the vortices be circular, any one of them, if alone, moves with a constant velocity along its axis, at right angles to the plane of the circle, in the direction of the motion of the fluid rotating on the inner surface of the ring.

(8) The fluid encircled by the ring moves along its axis in the direction of the motion of the ring, and with a greater velocity.

(9) If two circular vortices move along the same axis, one following the other, the one in the rear moves faster, and diminishes in diameter; the one in advancc

moves slower, and increases in diameter. If the strength and size of the two are nearly equal, the one in the rear overtakes the other, and passes through it. The two now having changed places, the action is repeated indefinitely.

(10) If two circular vortices of equal strength move along the same axis toward one another, the velocities of both gradually decrease and their diameters increase. The same result follows if one such vortex moves toward a solid barrier.

The preceding statements apply only to vortices set up in a perfect fluid. They may, however, be illustrated by experiment. To produce circular vortices in the air, we use a box which has one of its ends flexible. A circular opening is cut in the opposite end. The box is filled with smoke or with finely divided sal-ammoniac, resulting from the combination of the vapors of ammonia and hydrochloric acid. On striking the flexible end of the box, the rings are at once sent out.

The ring is easily seen to be made up of particles revolving about a central core in the form of a ring. With such rings many of the preceding statements may be verified.

An illustration of the open vortex is seen when an oar-blade is drawn through the water. By making such open vortices, using a circular dish, many of the observations with the smoke-rings may be repeated in another form.

85. ***Air-Pumps.***—The fact that gases, unlike liquids, are easily compressed, and obey Boyle's law under ordinary conditions of temperature and pressure, underlies the construction and operation of several pieces of ap-

paratus of great value in physical investigations. The most important of these is the air-pump.

The working portion of the air-pump is constructed essentially like the common lifting-pump already described. The valves must be light and accurately fitted. The vessel from which the air is to be exhausted is joined to the pump by a tube, the orifice of which is closed by the valve in the bottom of the cylinder.

A special form of vessel much used in pneumatic experiments is called the "receiver." It is usually a glass cylinder, open at one end, and closed by a hemispherical portion at the other. The edge of the cylinder at the open end is ground perfectly true, so that all points in it are in the same plane. This ground edge fits upon a plane surface of roughened brass, or ground glass, called the "plate," through which enters the tube which joins the receiver to the cylinder of the pump. The joint between the receiver and the plate is made tight by a little oil or vaseline.

The action of the pump is as follows: as the piston is raised, the pressure on the upper surface of the valve in the cylinder is diminished, and the air in the vessel expands in accordance with Boyle's law, lifts the valve, and distributes itself in the cylinder, so that the pressure at all points in the vessel and the cylinder is the same. The piston is now forced down, the lower valve is closed by the increased pressure on its upper surface, the valve in the piston is opened, and the air in the cylinder escapes. At each successive stroke of the pump this process is repeated, until the pressure of the remnant of air left in the vessel is no longer sufficient to lift the valves.

The density of the air left in the vessel after a given number of strokes, is determined, provided there be no leakage, by the relations of the volumes of the vessel and the cylinder.

Let V represent the volume of the vessel, and C that of the cylinder when the piston is raised to the full extent of the stroke. Let d and d_1 respectively represent the density of the air in the vessel before and after one stroke has been made. After one down and one up stroke has been made, the air which filled the volume V now fills $V + C$. It follows that

$$\frac{d_1}{d} = \frac{V}{V+C}.$$

As this ratio is constant no matter what density may be considered, it follows, that, if d_n represent the density after n strokes,

$$\frac{d_n}{d} = \left(\frac{V}{V+C}\right)^n. \qquad (37)$$

As this fraction cannot vanish until n becomes infinite, it is plain that a perfect vacuum can never, even theoretically, be obtained by means of the air-pump. If, however, the cylinder be large, the fraction decreases rapidly, and a few strokes are sufficient to bring the density to such a point that either the pressure is insufficient to lift the valves, or the leakage through the various joints of the pump counterbalances the effect of longer pumping.

In the best air-pumps the valves are made to open automatically. In Fig. 43 is represented one of the methods by which this is accomplished. They can then be made heavier and with a larger surface of con-

tact, so that the leakage is decreased, and the limit of the useful action of the pump is much extended. With the best pumps of this sort a pressure of one-half a millimetre of mercury is reached.

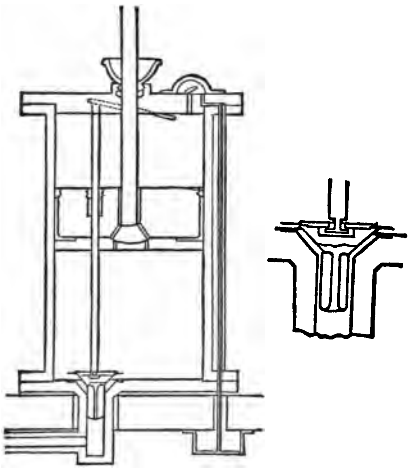

Fig. 43.

The Sprengel air-pump depends for its action upon the principle discussed in § 83, — that a stream of liquid running down a cylinder diminishes the pressure upon its walls. In the Sprengel pump the liquid used is mercury. It runs from a large vessel down a glass tube, into the wall of which, at a distance from the bottom of the tube of more than 760 millimetres, enters the tube which connects with the receiver. The lower end of the vertical tube dips into mercury, which prevents air from passing up along the walls of the tube. When the stream of mercury first begins to flow, the air enters the column from the receiver, in consequence

of the diminished pressure, passes down with the mercury in large bubbles, and emerges at the bottom of the tube. As the exhaustion proceeds, the bubbles become smaller and less frequent, and the mercury falls in the tube with a sharp, metallic sound. It is evident, that, as in the case of the ordinary air-pump, there cannot be a perfect vacuum secured. There being no leakage, however, in this form of the air-pump, a very high degree of exhaustion can be reached.

The Morren or Alvergniat mercury-pump is in principle merely a common air-pump, in which combinations of stop-cocks are used instead of valves, and a column of mercury in place of the piston. Though slow in its operation, it is sometimes used, its particular excellence being that there need be scarcely any leakage.

The compressing pump is used, as its name implies, to increase the density of air or any other gas within a receiver. The receiver in this case is generally a strong metallic vessel. The working parts of the pump are precisely those of the air-pump, with the exception that the valves open downwards. As the piston is raised, air enters the cylinder, and is forced into the receiver at the down-stroke.

86. ***Manometers.***—The manometer is an instrument used for measuring pressures. One variety depends for its performance upon the regularity of change of volume of a gas with change of pressure. This in its typical form consists of a heavy glass tube of uniform bore, sealed at one end, with the open end immersed in a basin of mercury. The pressure to be measured is applied to the surface of the mercury in the basin. As this pressure increases, the air contained in the tube

is compressed, and a column of mercury is forced up the tube. The top of this column serves as an index. We know, from Boyle's law, that, when the volume of air has diminished one-half, the pressure is doubled. The downward pressure of the mercury column makes up a part of this pressure; and the pressure acting on the surface of the mercury in the basin is greater than that indicated by the compression of the air in the tube, by the pressure due to the mercury column. For many purposes the manometer tube may be made very short, and the pressure of the mercury column that rises in it may be neglected.

87. ***Aneroids.*** — The aneroid is an instrument used to determine ordinary atmospheric pressures. On account both of its delicacy and its easy transportability, it is often used instead of the barometer. It consists of a metallic box, the cover of which is made of thin sheet-metal corrugated in circular grooves. The air is partially exhausted from the box, and it is then sealed. Any change in the pressure of the atmosphere causes the corrugated top to move. This motion is very slight, but is made perceptible, either by a combination of levers, which amplifies it, or by an arm rigidly fixed on the top, the motion of which is observed by a microscope. The indications of the aneroid are compared with a standard mercurial barometer, and an empirical scale is thus made, by means of which the aneroid may be used to determine pressures directly.

88. ***Limitations to the Accuracy of Boyle's Law.*** — In all the previous discussions, we have dealt with gases as if they obeyed Boyle's law with absolute exactness. This, however, is not the case. In the first place, some gases

at ordinary temperatures can be liquefied by pressure. As these gases approach more nearly the point of liquefaction, the product PV of the volume and pressure becomes less than it ought to be in accordance with Boyle's law.

Secondly, those gases which cannot be liquefied at ordinary temperatures by any pressure, however great, show a different departure from the law. For every gas, except hydrogen, there is a minimum value of the product PV. At ordinary temperatures and small pressures the gas follows Boyle's law quite closely, becoming, however, more compressible as the pressure increases, until the minimum value of PV is reached. It then becomes gradually less compressible, and at high pressures its volume is much greater than that determined by Boyle's law. If the temperature be raised, the agreement with the law is closer, and the pressure at which the minimum value of PV occurs is greater. Hydrogen seems to differ from the other gases, only in that the pressures at which the observations upon it were made were probably greater than the one at which its minimum value of PV occurs. The volume of the compressed hydrogen is uniformly greater than that required by Boyle's law.

Important modifications are introduced into the behavior of gases under pressure by subjecting them to intense cold. It is then found that all gases, without exception, can be liquefied, and even solidified. Though no observations have been made upon their adherence to Boyle's law under the conditions of the experiments, it is highly probable, that, as they approach the point of liquefaction, they behave as those gases which can be

liquefied at ordinary temperatures, and become more compressible than they should be according to the law.

The subject is intimately connected with the subject of "critical temperature," and will be again discussed in the chapters on "Heat."

BOOK II.

HEAT.

PRELIMINARY.

89. ***General Effects.*** — First, Bodies are warmed, or their temperature is raised, by heat. The sense of touch is often sufficient to show difference in temperature; but the true criterion is the transfer of heat from the warmer to the colder body when the two bodies are brought in contact, and no work is done by one upon the other. This transfer is known by some of the effects described below.

Second, Bodies, in general, expand when heated. Experiment shows that different substances expand differently for the same rise of temperature. Gases, in general, expand more than liquids, and liquids more than solids.

Expansion, however, does not universally accompany rise of temperature. A few substances contract when heated.

Third, Heat changes the state of bodies, as solids into liquids, and liquids into gases. The melting of ice and the conversion of water into steam are familiar examples.

Fourth, Heat breaks up chemical compounds. The compounds of sodium, potassium, lithium, etc., give to the flame of a Bunsen lamp the characteristic color of

the *metallic* vapor,—evidence that the heat has separated the metal from the element with which it was combined.

Fifth, Heat produces electric currents.

Sixth, Heat performs mechanical work. The heat produced in the furnace of the steam-boiler drives the engine.

90. ***Production of Heat.***—Heat is produced by the reverse of the above processes. Bodies that expand by being heated are warmed by compression. The fire-syringe illustrates this. Heat is generated when a liquid becomes solid or a gas becomes liquid. Experiments in proof of this will be adduced later.

Heat is generated when substances combine chemically. The heat of combustion is one example.

Heat is generated by electric currents. The flow of electricity through a conductor always generates heat, greater in quantity as the resistance of the conductor is greater. The electric light is the result of heat so produced.

Heat is produced by mechanical energy. Buildings are sometimes warmed by the heat produced by the friction of iron plates made to revolve under pressure by means of water-power. Mills have been set on fire by heat generated by the friction of the machinery not properly lubricated.

91. ***Nature of Heat.***—Heat was for a long time considered to be a substance which passed from one body to another, lowering the temperature of the one and raising that of the other, which combined with solids to form liquids, and with liquids to form gases or vapors. But the most delicate balances fail to show any change

of weight when heat passes from one body to another. Count Rumford was able to raise a considerable quantity of water to the boiling-point by the friction of a blunt boring-tool within the bore of a cannon. It was evident that the heat manifested in this experiment could not have come from any of the bodies present. It was evident, also, that heat would continue to be developed as long as the borer continued to revolve; that is, the supply of heat was practically inexhaustible. The heat must have been *generated* by the friction.

That ice is not melted by the combination with it of a heat substance was shown early in the present century by an experiment by Sir Humphry Davy. He caused ice to melt by friction of one piece upon another in a vacuum, the experiment being performed in a room where the temperature was below the melting-point of ice. There was no source from which heat could be drawn. The ice must, therefore, have been melted by the friction.

The explanation of such experiments is very simple upon the hypothesis that *heat is a form of energy.*

Count Rumford was convinced that the heat in his experiment was only transformed mechanical energy; but to demonstrate this it was necessary to prove that the quantity of heat produced was always proportional to the quantity of mechanical energy employed. This was done in the most complete manner by Joule in a series of experiments extending from 1842 to 1849. He showed, that, however the heat was produced by mechanical means, whether by the agitation of water by a paddle-wheel, the agitation of mercury, or the friction of iron plates upon each other, the same expenditure of

mechanical energy always developed the same quantity of heat. Joule also proved the perfect equivalence of heat and electrical energy.

There are many reasons for believing that "*heat is a mode of motion.*" Upon this hypothesis the molecules of all bodies are in constant motion. Upon the energy of this motion depends the temperature of the body. When a body is warmed, the motion of its particles becomes more energetic; when it is cooled, the energy of the motion diminishes.

CHAPTER I.

THERMOMETRY.

92. ***Temperature.*** — Two bodies are said to be at the same temperature, when, if they be brought into intimate contact, no heat is transferred from one to the other. A body is at a high temperature relatively to other bodies when it gives up heat to them. The fact that it gives up heat may be shown by its own contraction. A body is at a low temperature when it receives heat from surrounding bodies. It is understood, of course, in what is said above, that one body has no action upon the other; in other words, no work is done by one body upon the other when they are brought in contact.

93. ***Thermometers.*** — Experiments show, that, in general, bodies expand, and their temperature rises progressively, with the application of heat. An instrument may be constructed which will show at any instant the volume of a body selected for the purpose. If the volume increases, we know that the temperature rises; if the volume remains constant or diminishes, we know that the temperature remains stationary or falls. Such an instrument is a *thermometer*.

The thermometer most in use consists of a glass bulb with a fine tube attached, the bulb and part of the tube

containing mercury. In order that the thermometers of different makers may give similar readings, it is necessary to adopt two standard temperatures which can be easily and certainly reproduced. The temperatures adopted are the melting-point of ice, and the temperature of steam from boiling water under a pressure equal to that of a column of mercury 760 millimetres high at Paris. The instrument having been filled with mercury, it is plunged in melting ice, from which the water is allowed to drain away, and the place occupied by the top of the column is marked upon the stem. It is then placed in an apparatus consisting of a vessel in which water is boiled, the steam rising through a tube, surrounding the thermometer, and then descending by an annular space between the inner and outer tubes, and escaping at the bottom. The thermometer does not touch the water, but is entirely surrounded by steam. The point reached by the column of mercury is marked on the stem, as before. The space between these two points is now divided into a number of equal parts. But, while all makers of thermometers have adopted the same standard temperatures for the fixed points of the scale, they differ as to the number of divisions between these points. The thermometers used for scientific purposes, and in general use in France, have the space between the fixed points divided into a hundred equal parts or *degrees*, the melting-point of ice being marked 0°, and the boiling-point 100°. This is called the *Centigrade* or *Celsius* Scale.

The *Réaumur* Scale, in use in Germany, has eighty degrees between the freezing and boiling points, and the boiling-point is marked 80°.

The *Fahrenheit* Scale, in general use in England and America, has a hundred and eighty degrees between the freezing and boiling points; the former being marked 32°, and the latter 212°.

The following formulas represent the relations between these scales:—

t_c = temperature Centigrade.
t_f = temperature Fahrenheit.
t_r = temperature Réaumur.

$$\left.\begin{aligned} t_c &= \tfrac{5}{9}(t_f - 32) = \tfrac{5}{4}t_r. \\ t_f &= \tfrac{9}{5}t_c + 32 = \tfrac{9}{4}t_r + 32. \\ t_r &= \tfrac{4}{5}t_c = \tfrac{4}{9}(t_f - 32). \end{aligned}\right\} \qquad (38)$$

The divisions in all these cases are extended below the zero point, and are numbered from zero downward. Temperatures below zero must, therefore, be read and treated as negative quantities.

A few points in the process of construction of a thermometer deserve notice. It is found that glass, after being heated to a high temperature, does not return to its original volume for some time after being cooled. The bulb of a thermometer must be heated in the process of filling with mercury, and it will not return to its normal volume for some months. The construction of the scale should not be proceeded with till the reservoir has ceased to contract. For the same reason, if the thermometer is used for high temperatures, even the temperature of boiling water, time must be given for the reservoir to return to its original volume before using it for the measurement of low temperatures.

It is essential that the diameter of the tube should be nearly uniform throughout, and that the divisions of

the scale should represent equal capacities in the tube. To test the tube, a thread of mercury of say 50 millimetres length is introduced, and its length measured in different parts of the tube. If the length varies by more than a millimetre, the tube should be rejected. If the tube be found to be suitable, a reservoir is attached, mercury is introduced, and the tube sealed after the mercury has been heated to expel the air. When it is ready for graduation, the fixed points are determined; then a thread of mercury of about ten degrees in length is detached from the column, and its length measured in all parts of the tube. By reference to these measurements, the tube is so graduated that the divisions represent parts of equal capacity, and are not necessarily of equal length.

If such a thermometer indicate a temperature of 10° (Centigrade), this means that the thermometer is in such a thermal condition that the volume of the mercury has increased from zero one-tenth of its total expansion from zero to 100°. There is no reason for supposing that this represents the same proportional rise of temperature. If a thermometer be constructed in the manner described, using some liquid beside mercury, it would not in general indicate the same temperature as the mercurial thermometer, except at the two standard points. It is plain, therefore, that a given fraction of the expansion of a liquid from zero to 100° cannot be taken as representing the same fraction of the rise of temperature.

94. ***Air Thermometer.***—If a gas is heated, and its volume kept constant, its pressure increases. The so-called permanent gases—those that are liquefied only

with great difficulty — are each found to increase in pressure by almost exactly the same amount for the same increase of temperature. This is itself a reason for supposing that the increase of pressure is proportional to the increase of temperature. There are theoretical reasons, as will be seen later, for the same supposition.

If air be substituted for mercury in the thermometer, and means are provided for maintaining its volume constant, and measuring its pressure, the instrument becomes an *air thermometer.* For the reasons given above, the air thermometer is taken as the standard instrument for scientific purposes. Its use, however, involves several careful observations and tedious computations. It is, therefore, mainly employed as an instrument with which to compare other instruments. By making such a comparison, and constructing a table of corrections, the readings of any thermometer may be reduced to the corresponding readings of the air thermometer.

95. ***Limits in the Range of the Mercurial Thermometer.*** — The range of temperature for which the mercurial thermometer may be employed is limited by the freezing of the mercury on the one hand, and its boiling on the other. For temperatures below the freezing-point of mercury, alcohol thermometers may be employed. For the measurement of high temperatures, several different instruments have been made use of. One depends upon the expansion of a bar of platinum, another upon the variation in the electrical resistance of platinum wire, another upon the strength of the electric current generated by a thermo-electric pair; another consists of

a globe of glass or porcelain, having a short tube with a small orifice containing a small quantity of mercury. The whole is placed in the furnace or other place whose temperature is to be measured. The mercury boils, the air and the excess of mercury are expelled, and we have finally the globe full of mercury vapor at the temperature of the furnace. The globe is cooled, and the weight of the mercury left in it determined. Knowing this and the volume of the globe, the temperature can be computed.

96. ***Registering Thermometers.*** — Maximum and minimum thermometers are employed to register the highest and lowest temperatures reached during a given period. By a change in construction, the ordinary mercury becomes a self-registering maximum thermometer. This change consists in contracting the tube just above the reservoir to such an extent, that, though the mercury is pushed through it as the temperature rises, it does not return as the temperature falls. It thus serves as an index to show the highest temperature reached during the period of its exposure. After an observation has been made, the thermometer is re-adjusted for a new observation by allowing the instrument to swing out of the horizontal position in which it usually rests, about a point near the upper extremity of the tube.

In the construction of the minimum thermometer, alcohol is the liquid employed. Before sealing, an index of glass, smaller in diameter than the bore of the tube, is inserted. When the instrument is adjusted for use, this is brought in contact with the extremity of the column, and the tube is placed in a horizontal position. If, now, the alcohol expand, it will flow past the index

without moving it; but, if it contract, it will, by adhesion, draw the index after it. Thus the minimum temperature is registered.

Registering thermometers have been made to give a continuous record of changes of temperature. One method of effecting this is to produce an image of the thermometer tube, which is strongly illuminated by a light placed behind it, upon a screen of sensitized paper which moves continuously by means of clockwork. Light is excluded from the whole of the paper, except the part that corresponds to the image of the tube *above the mercury*. This part of the paper is blackened by the light; and, as the paper moves, the edge of the blackened portion will present a sinuous line corresponding to the movements of the mercury of the thermometer.

CHAPTER II.

CALORIMETRY.

97. ***Unit of Heat.*** — It is evident that more heat is required to raise the temperature of a large quantity of a substance t degrees than to raise the temperature of a small quantity of the same substance by the same amount. More heat is generated by a gas-flame in one hour than in half an hour. More heat is generated by burning a ton of coal than by burning a less amount. Heat is therefore a quantity the magnitude of which may be expressed in terms of some convenient unit. The unit of heat generally adopted is the heat required to raise the temperature of one kilogram of water from zero to one degree. It is called a *calorie.*

It is sometimes convenient to employ a smaller unit, — the quantity of heat necessary to raise one gram of water from zero to one degree. This unit is designated the *lesser calorie.* It is one one-thousandth of the larger unit. It may, therefore, be called a *mille-calorie.*

98. ***Heat required to raise the Temperature of a Mass of Water.*** — It is evident that to raise the temperature of m kilograms of water from zero to one degree will require m calories. If the temperature of the same quantity of water fall from one degree to zero, the same quantity of heat is given to surrounding bodies.

Experiment shows, that, if the same quantity of water be raised to different temperatures, quantities of heat *nearly* proportional to the rise in temperature will be required: hence, to raise the temperature of m kilograms of water from zero to t degrees requires mt calories very nearly. This is shown by mixing water at a lower with water at a higher temperature. The temperature of the mixture will be almost exactly the mean. Regnault, trying this experiment with the greatest care, found the temperature of the mixture a little higher than the mean, and concluded that the quantity of heat required to raise the temperature of a kilogram of water one degree increases slightly with the temperature; that is, to raise the temperature of a kilogram of water from twenty to twenty-one degrees requires a little more heat than to raise the temperature of the same quantity of water from zero to one degree.

Rowland found, by mixing water at various temperatures, and also by measuring the energy required to raise the temperature of water by agitation by a paddle-wheel, that, when the air thermometer is taken as a standard, the quantity of heat necessary to raise the temperature of a given quantity of water one degree diminishes slightly from zero to thirty degrees, and then increases to the boiling-point.

99. ***Specific Heat.*** — Only one-thirtieth as much heat is required to raise the temperature of a kilogram of mercury from zero to one degree as to raise the temperature of a kilogram of water through the same range. Other substances require, to raise their temperature, quantities of heat peculiar to each substance.

The quantity of heat required to raise the temperature of one kilogram of a substance from zero to one degree is called its *specific heat.*

The quantity of heat required to raise one kilogram of a substance from t to $t+1$ degrees is called its specific heat at temperature t.*

The specific heats of substances are generally nearly constant between zero and one hundred degrees. The *mean specific heat* of a substance between zero and one hundred degrees is the one usually given in the tables.

The measurement of specific heat is one of the important objects of calorimetry.

100. ***Ice Calorimeter.*** — The ice calorimeter of Black consisted of a block of pure ice having a cavity in its interior covered by a thick slab of ice. The body whose specific heat was to be determined was heated to T degrees, then dropped into the cavity, and immediately covered by the slab. After a short time the temperature of the body falls to zero, and in doing so converts a certain quantity of ice into water. This water is removed by a sponge of known weight, and its weight is determined. It will be shown, that to melt a kilogram of ice requires 80 calories; if, then, the weight of the body be P, and its specific heat x, it gives up, in falling from T degrees to zero, PxT calories. On the other hand, if p kilograms of ice be melted, the heat required is $80p$. Therefore $PxT = 80p$.

$$x = \frac{80p}{PT}. \tag{39}$$

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* Or, more exactly, the ratio of the small quantity of heat dQ to the corresponding rise of temperature dt is the specific heat at t degrees.

Bunsen's ice calorimeter (Fig. 44) is used for determining the specific heats of substances of which only a small quantity is at hand. The apparatus is entirely of glass. The tube *B* is filled with water and mercury, the latter extending into the graduated capillary tube *C*. To use the apparatus, alcohol which has been artificially cooled to below zero is passed through the tube *A*, causing a layer of ice to form around the outside of this tube. As water freezes, it expands. This causes the mercury to advance in the capillary tube *C*. When a sufficient quantity of ice has been formed, the alcohol is removed from *A*, the apparatus is surrounded by melting snow or ice, and a small quantity of water is introduced, which soon falls in temperature to zero. The position of the mercury in *C* is now noted; and the substance, at the temperature of the surrounding air — say, fifteen or twenty degrees — is dropped into the water in *A*. Its temperature quickly falls to zero, its excess of heat being entirely employed in melting the ice which surrounds the tube *A*. As the ice melts, the mercury in the tube *C* retreats. The change of position is an indication of the quantity of ice melted, and the quantity of ice melted measures the heat given up by the substance. The number of divisions of the tube *C* corresponding to one calorie can be determined by direct experiment. Perform the experiment as described above,

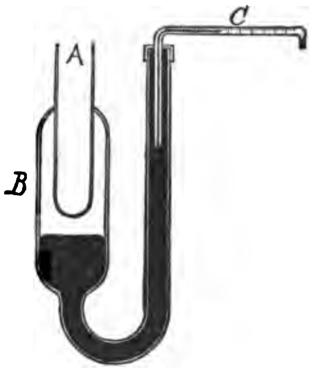

Fig. 44.

using a substance whose specific heat c is known. If its mass be p and its temperature t, it gives up, in cooling to zero, cpt calories. If the mercury retreat at the same time n divisions, to one division corresponds $\frac{cpt}{n}$ calories. If, now, a mass p' of a substance at a temperature t' be introduced, and the mercury fall n' divisions, the number of calories which must have been given to the ice is $\frac{n'cpt}{n}$, and the specific heat of the substance

$$x = \frac{n'cpt}{np't'}. \tag{40}$$

101. ***Method of Mixtures.***—This consists in bringing together at different temperatures the substance whose specific heat is desired and another whose specific heat is known, and noting the change of temperature which each undergoes.

The water calorimeter consists of a vessel of very thin copper or brass, highly polished, and placed within another vessel upon non-conducting supports. A mass P of the substance whose specific heat is to be determined is brought to a temperature T in a suitable bath, then plunged in water whose temperature is t, contained in the calorimeter. The whole will soon come to a common temperature θ.

The substance loses $Px(T - \theta)$ calories.

The heat gained by the calorimeter consists of the heat gained by the water, $p(\theta - t)$; heat gained by the vessel, $p'c'(\theta - t)$; heat gained by the glass-stirrer and the glass of the thermometer, $p''c''(\theta - t)$; heat gained by the mercury of the thermometer $p'''c'''(\theta - t)$: where

p represents the mass of the water, p' the mass of the vessel, p'' the mass of the glass of the stirrer and of the thermometer, p''' the mass of the mercury of the thermometer, c' the specific heat of the material of the vessel, c'' the specific heat of glass, and c''' the specific heat of mercury. If no heat is lost or gained by radiation, the heat lost by the substance is equal to that gained by the calorimeter: whence $Px(T-\theta) = (p + p'c' + p''c'' + p'''c''')(\theta - t)$.

To determine x from this formula, c', c'', and c''', must be known. Approximate values of these may be obtained and used in the formula, but it is better to determine the value of $p'c' + p''c'' + p'''c''' = p_1$ by experiment. Let a mass of water P, at a temperature T, be substituted for the substance whose specific heat was to be determined in the experiment described above. The equation will then become

$$P(T-\theta) = (p + p_1)(\theta - t), \qquad (41)$$

in which p_1 is the only unknown quantity. p_1 is the *water equivalent* of the calorimeter and accessories. It is determined, once for all, as just described.

There is a source of error in the use of the instrument, due to the radiation of heat during the experiment. This may be nearly compensated by making a preliminary experiment to determine what change of temperature the calorimeter will experience; then, for the final experiment, bring the calorimeter and its contents about half that amount below the temperature of the surrounding air. The calorimeter will then receive heat from the surrounding medium during the first part of the experiment, and lose heat during the second part.

The rise of temperature is, however, much more rapid at the beginning than at the end of the experiment. This rise from the initial temperature to the temperature of the surrounding medium occupies less time than the rise from the latter to the final temperature. The gain of heat, therefore, does not exactly compensate for the loss. If greater accuracy be required, the rate of cooling of the calorimeter must be determined by putting into it warm water, the same in quantity as would be used in experiments for determining specific heat, and noting from minute to minute its temperature. Such an experiment furnishes the data for computing the loss or gain by radiation. To secure accurate results the body must be transferred from the bath to the calorimeter without sensible loss of heat.

102. ***Method of Comparison.***—This consists in conveying to the substance whose specific heat is to be determined, a known quantity of heat, and comparing the consequent rise of temperature with that produced by the same amount of heat in a substance whose specific heat is known. In the early attempts to use this method, the heat produced by the same flame burning for a given time was applied successively to different liquids. A more exact method was the combustion, within the calorimeter, of a known weight of hydrogen; but the best method of obtaining a known quantity of heat is by means of an electric current of known strength flowing through a wire of known resistance wrapped upon the calorimeter.

103. ***Method of Cooling.***—This consists in noting the time required for the calorimeter, in a space kept constantly at 0°, to cool from a temperature t' to a tempera-

ture t, first, when empty; second, when containing a given weight of water; third, when containing a given weight of the substance whose specific heat is sought. The thermo-calorimeter of Regnault, represented in Fig. 45, is an example. It consists of an alcohol thermometer, having its bulb A enlarged and made in the form of a hollow cylinder, inside of which the substance is placed. The thermometer is warmed, then placed in a vessel surrounded by melting ice. It radiates heat to the sides of the vessel, and the alcohol in the tube falls. Let x be the time occupied in falling from division n to division n' when the space B is empty. Let the time between the same two divisions when the space B contains a mass P of water, and when it contains a mass P' of the substance whose specific heat c' is sought, be respectively x' and x''. Let M be the water equivalent of the instrument. We then have

Fig. 45.

$$\frac{M}{x} = \frac{M + P}{x'} = \frac{M + P'c'}{x''},$$

since, under the conditions of the experiment, the heat lost per second must be the same in each case.

Eliminating M,

$$c' = \frac{P}{P'}\left(\frac{x'' - x}{x' - x}\right). \qquad (42)$$

104. ***Determination of the Mechanical Equivalent of Heat.*** — It has been seen, that to a certain amount of mechanical work corresponds always a certain fixed amount of

heat. The mechanical equivalent of heat is the energy in mechanical units corresponding to the unit of heat.

Heat applied to a body may increase the vibratory motion of its molecules; that is, add to their kinetic energy. It may perform internal work by moving the molecules against molecular forces. It may perform external work by producing motion against external forces. If we could estimate these effects in mechanical units, we should have the mechanical equivalent of heat. But the kinetic energy of the molecules cannot be estimated, for we do not know their mass nor their velocity. We must, therefore, in the present state of our knowledge, resort to direct experiment to determine the heat equivalent. The experiments of Joule have already been referred to (p. 157). A paddle-wheel is made to revolve by means of weights in a vessel filled with water. In this vessel are stationary wings, to prevent the water from acquiring a rotary motion with the paddle-wheel. By the revolution of the wheel the water is warmed. The heat so generated is estimated from the rise of temperature, while the mechanical energy required to produce it is given by the fall of the driving-weight. Joule repeated these experiments, substituting mercury for the water. In another experiment he substituted an iron plate for the paddle-wheel, and made it revolve with friction upon a fixed iron plate under water.

The results of Joule's experiments were, when

Water was used.	423.9
Mercury was used	425.7
Iron was used	426.1

kilogram-metres per calorie. Joule gives the preference

to the smaller value, and it has been generally accepted as the mechanical equivalent. This mechanical equivalent is called Joule's equivalent, and is represented by *J*. In absolute units it is 41,595,000,000 ergs per calorie.

Rowland has repeated Joule's experiment of the paddle-wheel in water; but he caused the paddle-wheel

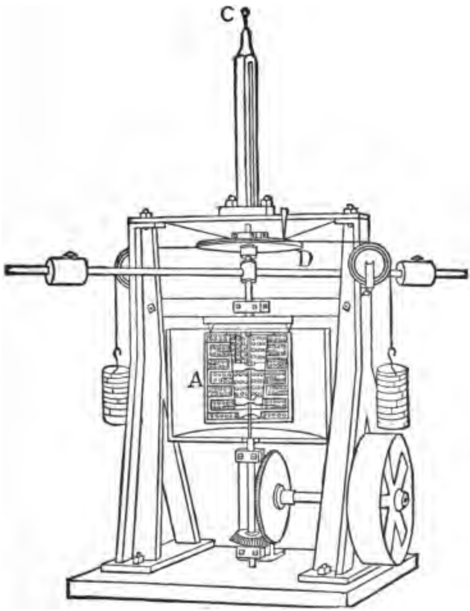

Fig. 46.

to revolve by means of an engine, and determined the moment of the couple required to prevent the revolution of the calorimeter. Fig. 46 shows the apparatus. The shaft of the paddle-wheel projects through the bottom of the calorimeter, and is driven by means of a bevel-gear. The vessel *A* is suspended from *C* by a

torsion wire, and its tendency to rotate balanced by weights attached to cords which act upon the circumference of a pulley *D*. By this disposition of the apparatus he was able to consume about one-half horse-power in the calorimeter, and obtain a rise of temperature of 35° per hour; while in Joule's experiments the rise of temperature per hour was less than 1°. These experiments give, for the mechanical equivalent of one kilogram-degree at 5°, 429.8 kilogram-metres; at 20°, 426.4 kilogram-metres.

Several other methods have been employed for determining the mechanical equivalent. The concordance of the results by all these methods is sufficient to warrant the statement that to a given amount of mechanical work always corresponds the same amount of heat.

CHAPTER III.

EFFECTS OF HEAT.—SOLIDS.—LIQUIDS.

105. ***Expansion of Solids.***—When heat is applied to a body, it increases the kinetic energy of the molecules (raises the temperature), and increases the potential energy, by forcing the molecules farther apart against their mutual attractions and any external forces that may resist expansion. Since the internal work to be done when a solid or liquid expands varies greatly for different substances, it would be expected that the amount of expansion for a given rise of temperature would vary greatly.

In studying the expansion of solids, we distinguish *linear* and *voluminal* expansion.

The increase which occurs in the unit length of a substance for a rise of temperature from zero to 1° C. is called the co-efficient of expansion. Experiment shows that the expansion for one degree is very nearly constant between zero and 100°. Hence the formulas,—

Let l_0 = length of the body at 0°.
Let l_t = length of the body at t°.
Let $l_{t'}$ = length of the body at t'°.
Let a = co-efficient of expansion.

A length l_0 will expand l_0a for a rise of one degree in

temperature, and tl_0a for a rise of $t°$. Its length after expansion is

$$l_t = l_0(1 + at); \qquad (43)$$

hence

$$l_0 = \frac{l_t}{1 + at},$$

or, approximately, $l_0 = l_t(1 - at)$, as will be seen by performing the division.

The binomial $1 + at$ is called the factor of expansion.

In the same way, if k equal co-efficient of voluminal expansion, the volume of a body at $t°$ will be

$$V_t = V_0(1 + kt); \qquad (44)$$

and, since density is inversely as the volume,

$$d_t = \frac{d_0}{1 + kt}. \qquad (45)$$

For a homogeneous solid, the co-efficient of voluminal expansion is three times that of linear expansion; for if a cube, the length of whose edge is unity, is raised one degree in temperature, its edge becomes $1 + a$ and its volume $1 + 3a + 3a^2 + a^3$. Since a is very small, its square and cube may be neglected; and its volume at one degree is, therefore, $1 + 3a$. $3a$ is, therefore, the co-efficient of voluminal expansion.

106. ***Measurement of Co-efficients of Linear Expansion.*** — Co-efficients of linear expansion are measured by comparing the lengths, at different temperatures, of a bar of the substance whose co-efficient is required, with the length, at constant temperature, of another bar. The constant temperature of the latter bar is secured by

immersing it in melting ice. The bar whose co-efficient is sought may be carried to different temperatures by immersing it in a liquid bath; but it is found better to place the bar upon the instrument by means of which the comparisons are to be made, and leave it for several hours exposed to the air of the room, which is kept at a constant temperature by artificial means. Of course several hours must elapse between any two comparisons by this method, and its application is restricted to such ranges of temperature as may be obtained in occupied rooms; but within this range the observations can be made much more accurately than where the bar is immersed in a bath, and it is within this range that an accurate knowledge of co-efficients of expansion is of most importance.

107. ***Expansion of Liquids.***—In studying the expansion of a liquid, it is important to distinguish its *absolute* expansion, or the real increase in volume, and its *apparent* expansion, or its increase in volume in comparison with that of the containing vessel.

108. ***Absolute Expansion of Mercury.***—A knowledge of the co-efficients of expansion of mercury is of the greatest importance in physical investigations, since mercury is made use of for so many purposes in physical research. Various experimenters have occupied themselves with this problem, but Regnault has made the most elaborate and accurate experiments.

To determine the absolute co-efficient, the experiment must be so made that the expansion of the vessel shall not influence the result.

Regnault's experiment consisted in comparing the lengths of two columns of mercury, which produced

the same pressure, though of different temperatures. Two vertical tubes, *ab*, *a'b'* (Fig. 47), were connected at the top by a horizontal tube *aa'*, and at the bottom by a tube *bcdd'c'b'*, a part of which was of glass, and shaped like an inverted **U**. The top of the inverted **U** was connected by a tube *e* with a vessel *f*, in which air could be maintained at any desired pressure. Filling these tubes with mercury, it flows freely from one to the other at the top; but the flow of mercury between them at the bottom is prevented by air imprisoned in the inverted **U**, while the pressure is transmitted undiminished. The pressure at each end of the imprisoned air must, therefore, be the same; and, since *a* and *a'* are connected by a horizontal tube, the pressures at those points are the same also: hence the difference of pressure between *a* and *d* must be equal to the difference between *a'* and *d'*; and, from "Mechanics," § 76, it follows that the lengths of the columns (without regard to the diameters of the tubes) producing this difference, are inversely proportional to their densities. If, now, one branch be raised to temperature *t*, while the other remains at 0, the mercury in the inverted **U** will assume different levels. Measuring the height of each column from the surface of the mercury in the **U** to the horizontal tube at top, we have,

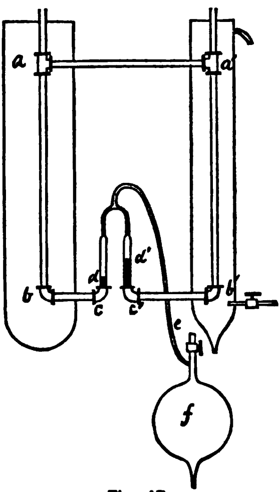

Fig. 47.

formula (45), if h and h' equal the height of the cold and warm columns respectively,

$$\frac{h}{h'} = \frac{d'}{d} = \frac{1}{1+kt},$$

$$hkt = h' - h,$$

$$k = \frac{h'-h}{ht}. \tag{46}$$

109. ***Apparent Expansion of Mercury.***—If a glass bulb (Fig. 48), furnished with a capillary tube, be filled with mercury at 0°, and heated to t°, some of the mercury runs out. The amount which overflows, evidently depends upon the difference of expansion between the mercury and the glass. Let P equal mass of mercury that fills the bulb and tube at 0°. After heating, there remains in the bulb a mass $P-p$ of mercury, which at 0° occupies the volume ab. The mercury which runs over, p, would at the same temperature fill the remainder of the bulb and tube. Volume $ab = \frac{P-p}{d}$, where d equals the density of mercury. Volume above $b = \frac{p}{d}$. The mercury in ab, when heated to t°, just fills the tube; and its apparent volume is $\frac{P}{d}$. If κ equal co-efficient of apparent expansion,

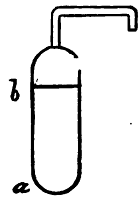

Fig. 48.

$$\frac{P}{d} = \frac{P-p}{d}(1+\kappa t), \tag{47}$$

or

$$\kappa = \frac{p}{(P-p)t}.$$

Having found κ, the instrument may be used as a thermometer; for, suppose it filled at 0°, and subjected to an unknown temperature t_1, we shall have, if p_1 equal the mercury that then runs over,

$$P=(P-p_1)(1+\kappa t_1),$$

$$t_1=\frac{p_1}{(P-p_1)\kappa}. \qquad (48)$$

The instrument is therefore called a weight thermometer.

The difference between the value κ, found above, and the absolute co-efficient k, is due to the expansion of the glass. And if k' be the co-efficient for glass,

$$k'=k-\kappa;$$

for, referring again to Fig. 48, the volume of the glass at 0° is $\frac{P}{d}$, and at $t°$ is $\frac{P}{d}(1+k't)$, which, from (47), equals

$$\frac{P-p}{d}(1+\kappa t)(1+k't).$$

The real volume at $t°$ of the mercury remaining in the tube is

$$\frac{P-p}{d}(1+kt).$$

Hence

$$(1+\kappa t)(1+k't)=(1+kt),$$
$$(1+\kappa t+k't+\kappa k't^2)=1+kt.$$

Since κ and k' are small quantities, their product may be neglected: hence

$$k'=k-\kappa. \qquad (49)$$

110. ***Determination of Voluminal Expansion of Solids.***—The weight thermometer may be used to determine the co-efficient of expansion of solids. For this purpose, the solid, whose volume at zero is known, must be introduced into the bulb by the glass-blower. If the bulb containing the solid be filled with mercury at 0°, and afterward heated to $t°$, it is evident that the amount of mercury that will overflow will depend upon the co-efficient of expansion of the solid, and upon the apparent co-efficient of mercury. The latter having been determined for the kind of glass used, the former can be deduced. By this means the voluminal co-efficients of some solids have been determined; and the results are found to verify the fact, deduced from theory (§ 105), that the voluminal co-efficient is three times the linear.

111. ***Absolute Expansion of Liquids other than Mercury.***—The weight thermometer may also serve to determine the co-efficients of expansion of liquids other than mercury; for, k' having been found as described above, the instrument may be filled with the liquid whose co-efficient is desired, and the apparent expansion of this liquid found exactly as was that of mercury. The absolute co-efficient for the liquid is then the sum of the apparent co-efficient and the co-efficient for the glass.

112. ***Expansion of Water.***—The use of water as a standard with which to compare the densities of other substances makes it necessary that we know, not merely its mean co-efficient of expansion, but its actual expansion, degree by degree. This is the more important since water expands very irregularly. The best series of determinations of the volumes of water at different temperatures are those of Matthiessen, who weighed

in water a mass of glass whose co-efficient of expansion had been previously determined.

Water contracts, instead of expanding, from 0° to 4°: from that temperature to its boiling-point it expands.

113. ***Determination of Specific Gravity.*** — Water at its maximum density at 4° is the standard to which is referred the specific gravities of solids and liquids. Since it is seldom practicable to make the determinations at that temperature, corrections must be made as follows: —

Let Δ_{tt} be the density of a substance at $t°$ compared to water at the same temperature. Let Δ_t be the density of the substance at $t°$ compared to water at 4°. Then $\Delta_t = \Delta_{tt} \times d_t$, where d_t is the density of water at $t°$; for, if

W = mass of substance,
W_t = mass of an equal volume of water at $t°$,
W_4 = mass of same volume of water at 4°,

$$\Delta_{tt} = \frac{W}{W_t},$$

$$d_t = \frac{W_t}{W_4},$$

$$\Delta_t = \frac{W}{W_4} = \frac{W}{W_t} \times \frac{W_t}{W_4} = \Delta_{tt} \times d_t.$$

If Δ_0 equals the density of a substance at 0°, $\Delta_0 = \Delta_t(1 + at)$, where a equals the co-efficient of voluminal expansion of the substance.

114. ***Convection.*** — If a vessel of water be heated at the bottom, the bottom layers become less dense than those above, producing a condition of instability. The lighter liquid is forced to rise, and the heavier layers from

above, coming to the bottom, are in their turn heated; causing continuous currents. This process is called convection. By this process, masses of liquids, although poor conductors, may be rapidly heated. Water is often heated in a reservoir at a distance from the source of heat by the circulation produced in pipes leading to the source of heat and back. The winds and the great currents of the ocean are convection currents. An interesting result follows from the fact that water has a maximum density. When the water of lakes cools in winter, currents are set up, and maintained as long as the surface water becomes more dense by cooling, or until the whole mass reaches 4°. Any further cooling makes the surface water lighter,—it remains at the surface, and its temperature rapidly falls to the freezing-point,—while the great mass of the water remains at the temperature of its maximum density.

115. ***Conduction.***—If one end of a metal rod be heated, it is found that the heat travels along the rod; since those portions at a distance from the source of heat finally become warm. This process of transmission of heat from molecule to molecule of a body, while the molecules themselves retain their relative places, is called conduction.

116. ***Flow of Heat across a Wall.***—To study the transfer of heat by this process, we will consider what takes place in a wall of homogeneous material, whose exposed surfaces, assumed to be of indefinite extent, are maintained at a constant difference of temperature. Suppose the wall to be cut by a series of planes parallel to the exposed surfaces: there must in the fixed state of the body be the same flow of heat across all sections; also

there must be a uniform fall of temperature from one side of the wall to the other, — that is, if $t' - t$ is the difference of temperature between the two exposed surfaces, and d the thickness of the wall, $\frac{t' - t}{d}$ is the fall of temperature per unit thickness, and $t' - \frac{t' - t}{d} d'$ is the temperature at a distance d' from the warmer surface.

To demonstrate this, suppose A, Fig. 49, be one exposed surface at temperature t', and B the other surface at temperature t: let a, a', a'', be three surfaces parallel to the faces of the wall, and at very small equal distances from each other. Suppose the temperatures to exist according to the law stated in the proposition: then the difference of temperature between a and a' will be the same as between a' and a''. Experiment shows that the flow of heat between two points in a body depends chiefly upon the *difference* of temperature, the distance between them, and the nature of the material. It is also found to depend, to a very limited extent, upon the temperature itself. The effect of this factor for the pairs of surfaces considered may, however, be neglected, since they are nearly at the same temperature. The other factors being the same for both pairs, it follows that there will be the same flow of heat from a' to a'' as from a to a'. The same will hold true for any other set of surfaces parallel to the faces of the wall: hence the molecules in any surface such as a' receive and part with equal

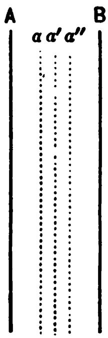

Fig. 49.

amounts of heat, and can neither rise nor fall in temperature. Hence the temperatures, once established in accordance with the law enunciated, could never change. On the other hand, if the difference of temperature between a and a' were greater than between a' and a'', the molecules in a' would receive more heat than they part with: their temperature would rise, and tend to equalize these differences. The proposition is therefore demonstrated.

117. ***Flow Proportional to*** $\frac{t'-t}{d}$. — It can be further shown that the flow of heat across walls of the same material is directly proportional to the differences of temperature, and inversely proportional to the thicknesses of the walls. Let there be two walls, A and B, of thickness d and δ, and let the temperatures of the exposed surfaces of A be t' and t, and of B, θ' and θ. Assume two planes in each wall parallel to the exposed surfaces, at very small equal distances apart, and similarly situated. The flow of heat in each wall, from the molecules in one plane to those in the other, will be proportional only to the differences of temperature, since all other things are equal. If ϵ be the common distance between the planes, $\frac{\epsilon}{d}(t'-t)$ will be the difference of temperature between the planes in A, and $\frac{\epsilon}{\delta}(\theta'-\theta)$ will be the same in B: hence

$$\frac{\textit{flow of heat between the planes in } A}{\textit{flow of heat between the planes in } B}=\frac{\frac{\epsilon}{d}(t'-t)}{\frac{\epsilon}{\delta}(\theta'-\theta)}=\frac{\frac{t'-t}{d}}{\frac{\theta'-\theta}{\delta}},$$

which proves the proposition, since the heat which flows

across the wall is the same as flows between any two planes.

It will be seen that $\frac{t'-t}{d}$ is the rate of fall of temperature at the section considered; and it follows, finally, that the flow of heat across any section parallel to the exposed surfaces of a wall is proportional to the rate of fall of temperature at that section.

118. ***Conductivity.*** — If, now, we consider a prism extending across the wall, bounded by planes perpendicular to the exposed surfaces, the area of its exposed bases being A, the quantity of heat which flows in a time T through this prism may be represented by

$$Q = K\frac{t'-t}{d}AT, \qquad (50)$$

where K is a constant depending upon the material of which the wall is composed. K is the *conductivity* of the substance, and may be defined as the quantity of heat which in unit time flows through a section of unit area in a wall of the substance whose thickness is unity, when its exposed surfaces are maintained at a difference of temperature of one degree; or, in other words, it is the quantity of heat which in unit time flows through a section of unit area in a substance, where the *rate of fall of temperature* at that section is unity. In the above discussions the temperatures t' and t are taken as the actual temperatures of the surfaces of the wall. If the colder surface of the wall be exposed to air of temperature T, to which the heat which traverses it is given up, t will be greater than T. The difference will depend upon the quantity of heat which flows, and upon the facility with which the surface parts with heat.

119. ***Flow of Heat along a Bar.***—If a prism of a substance have one of its bases maintained at a temperature t, while the other base and the sides are exposed to air at a lower temperature, the conditions of uniform fall of temperature no longer exist, and the amount of heat which flows through the different sections is no longer the same; but the amount of heat that flows through any section is still proportional to the rate of fall of temperature at that section, and is equal to the heat which escapes from the portion of the bar beyond the section.

120. ***Measurement of Conductivity.***—The bar heated at one end furnishes the best means of measuring conductivity. In Fig. 50 let AB represent a bar heated at A. Let the ordinates aa', bb', cc', represent the excess of temperatures above the temperature of the air at the points from which they are drawn.

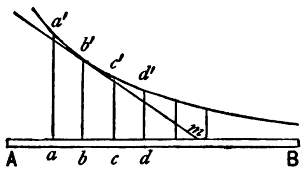

Fig. 50.

These temperatures may be determined by means of thermometers inserted in cavities in the bar, or by means of a thermopile. Draw the curve $a'b'c'd'$ . . . through the summits of the ordinates. The inclination of this curve at any point represents the rate of fall of temperature at that point. The tangent $b'm$, drawn to the point b', shows what would be the temperatures at various points of the bar if the fall were uniform and at the same rate as at b'. It shows, that, at the rate of fall at b', the bar would be at the temperature of the air at m; or, in the length bm, the fall of temperature would equal the amount represented by bb'. The rate of fall is, there-

fore, $\frac{bb'}{bm}$. If Q represent the quantity of heat passing the section at b in the unit time, we have, from (118),

$$Q = K \times \textit{rate of fall of temperature} \times \textit{area of section}.$$

Q is the quantity of heat that escapes from all that portion of the bar beyond b. It may be found by heating a short piece of the same bar to a high temperature, allowing it to cool in the same room under the same conditions that surround the bar AB, and observing its temperature from minute to minute as it falls. These observations furnish the data for computing the quantity of heat which escapes per minute from unit length of the bar at different temperatures. It is then easy to compute the amount of heat that escapes per minute from each portion, bc, cd, etc., of the bar beyond b; each portion being taken so short that its temperature throughout may, without sensible error, be considered uniform and the same as that at its middle point. Summing up all these quantities, we obtain the quantity Q which passes the section b in the unit time. Then

$$K = \frac{Q}{\textit{rate of fall of temperature at } b \times \textit{area of section}}.$$

121. ***Conductivity diminishes as Temperature rises.***— By the method described above, Principal Forbes determined the conductivity of a bar of iron at points b, c, d, etc., at different distances from the heated end, and found that the conductivity is not the same at all temperatures, but is greater as the temperature is lower.

122. ***Conductivity of Crystals.***—The conductivity of crystals of the regular system is the same in all direc-

tions, but in crystals of the other system it is not so. In a crystal of Iceland spar the conductivity is greatest in the direction of the axis of symmetry, and equal in all directions in a plane at right angles to that axis.

123. ***Conductivity of Non-homogeneous Solids.***—De la Rive and De Candolle were the first to show that wood conducts heat better in the direction of, than at right angles to, the fibres. Tyndall, by experimenting upon cubes cut from wood, has shown that the conductivity has a maximum value parallel to the fibres, a minimum value at right angles to the fibres and parallel to the annual layers, and a medium value at right angles to both fibres and annual layers. Feathers, fur, and the materials of clothing, are poor conductors because of their want of continuity.

124. ***Conductivity of Liquids.***—The conductivity of liquids can be measured in the same way as that of solids,—by noting the fall of temperature at various distances from the source of heat in a column of liquid heated at top. Great care must be taken in these experiments to avoid the existence of convection currents.

Liquids are generally poor conductors.

125. ***Radiation of Heat.***—We have now considered those cases in which there is a tendency to equalization of temperature between bodies in contact. The same tendency exists between bodies not in contact. This is effected by a process called radiation, which will be considered in another part of this work.

126. ***Effect of Variation of Temperature upon Specific Heat.***—It has already been stated (§ 98) that the specific heat of bodies changes with temperature. With

most substances specific heat increases as temperature rises.

The mean specific heat

Of iron

Between 0° and 100° is	0.1088
Between 100° and 200° is	0.1228

Of glass

Between 0° and 100° is	0.1770
Between 100° and 200° is	0.1900

Of the diamond the true specific heat

At 0° is	0.0947
At 50° is	0.1435
At 100° is	0.1905
At 200° is	0.2719

These examples are sufficient to show the fact stated.

127. ***Effect of Change of Physical State upon Specific Heat.*** — The specific heat of a substance is not the same in the solid and liquid states. In the solid state it is generally less than in the liquid. For example: —

	Mean Specific heat in the	
	Solid.	Liquid.
Water	0.504	1.000
Mercury	0.0314	0.333
Tin	0.056	0.0637
Lead	0.0314	0.0402

128. ***Atomic Heat.*** — Dulong and Petit have discovered the following law: *The product of the specific heat by the atomic weight of any simple body is a constant quantity.*

This is equivalent to saying, that *to raise the temperature of an atom of any simple substance one degree requires an amount of heat which is the same for all substances.*

The experiments of Regnault show that this law may be extended to compound bodies; that is, *for all compounds of similar chemical composition, the product of the total chemical equivalent by the specific heat is the same.*

The following table will illustrate the law of Dulong and Petit. The atomic weights are those given by Clarke.

ELEMENTS.	Specific heat of equal weights.	Atomic weight.	Product of specific heat into atomic weight.
Iron	0.114	55.9	6.372
Copper . . .	0.095	63.17	6.001
Mercury . .	0.0314 (solid)	199.71	6.128
Silver . . .	0.057	107.67	6.137
Gold	0.0329	196.15	6.453
Tin	0.056	117.7	6.591
Lead	0.0314	206.47	6.483
Zinc	0.0955	64.9	6.198

129. ***Fusion and Solidification.***—When ice at a temperature below zero is heated, its temperature rises to zero, then it begins to melt; and, however high the temperature of the medium that surrounds it, its temperature remains constant at zero as long as it remains in the solid state. This temperature is the *melting-point of ice*, and because of its fixity it is used as one of the standard temperatures in graduating thermometric scales. Other bodies melt at very different but at fixed

and definite temperatures, as shown in the following table:—

Substance.	Melting-point.
Carbonic acid	—78
Mercury	—39
Water	0
Phosphorus	44
Beeswax (white)	69
Sodium	90
Sulphur	111
Tin	228
Antimony	432
Silver	1000
Gold	1250
Platinum (about)	2000

Many substances cannot be melted, as they decompose by heat.

Alloys often melt at a lower temperature than either of their constituents. An alloy of lead 1, tin 1, bismuth 4, melts at 94°; while the lowest melting-point of its constituents is that of tin, 228°. An alloy of lead, tin, bismuth, and cadmium, melts at 62°.

If a liquid be placed in a medium whose temperature is below its melting-point, it will, in general, begin to solidify when its temperature reaches its melting-point, and it will remain at that temperature till it is all solidified. Under certain conditions, however, the temperature of a liquid may be lowered several degrees below its melting-point without solidification, as will be seen below.

130. ***Change of Volume with Change of State.***—Substances are generally more dense in the solid than in

the liquid state, but there are some notable exceptions. Water, on solidifying, expands: so that the density of ice at zero is only 0.9167, that of water at 4° being 1. This expansion exerts considerable force, as evidenced by the bursting of vessels and pipes containing water.

131. ***Change of Melting and Freezing Points.***—If water be enclosed in a vessel sufficiently strong to prevent its expansion, it cannot freeze except at a lower temperature. The freezing-point of water is, therefore, lowered by pressure. On the other hand, substances which contract on solidifying, have their solidification hastened by pressure.

The lowering of the melting-point of ice by pressure explains some remarkable phenomena. If pieces of ice be pressed together, even in warm water, they will be firmly united. Fragments of ice may be moulded, under heavy pressure, into a solid, transparent mass. This soldering together of masses of ice is called regelation. If a loop of wire be placed over a block of ice and weighted, it will cut its way slowly through the ice: but regelation will occur behind it; and, after passing through, the block will be found one solid mass, as before. The explanation of these phenomena is, that the ice is partially melted by the pressure: the liquid, as will be seen below, is colder than the ice; it finds its way to points of less pressure (for it is not to be supposed that over the entire surface between two masses of ice the pressure is the same); and there, because of its low temperature, it congeals, firmly uniting the two masses.

Water free from air, and kept perfectly quiet, will not

form ice at the ordinary freezing-point. Its temperature may be lowered to $-10°$ or $-12°$ before solidification commences. In this condition a slight jar, or the introduction of a small fragment of ice, will cause a sudden congelation of part of the liquid, accompanied by a rise in temperature in the whole mass to zero.

A similar phenomenon is observed in the case of several solutions, notably sodic sulphate and sodic acetate. If a saturated hot solution of one of these salts be made, and allowed to cool in a closed bottle in perfect quiet, it will not crystallize. Upon opening the bottle and admitting air, crystallization commences, and spreads rapidly through the mass, accompanied by a considerable rise of temperature. If the amount of salt dissolved in the water be not too great, the solution will remain liquid when cooled in the open air, and it may even suffer considerable disturbance by foreign bodies without crystallization; but crystallization begins immediately upon contact with the smallest crystal of the same salt.

132. ***Heat Equivalent of Fusion.***—Some facts that have appeared in the above account of the phenomena of fusion and solidification require further study. It has been seen, that, however rapidly the temperature of a solid may be rising, the moment fusion begins, the rise of temperature ceases. Whatever the heat to which a solid may be exposed, it cannot be made hotter than its melting-point.* When ice is melted by pressure, its temperature is lowered. When a liquid is cooled, fall

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* Some experiments of Carnelley, however, seem to indicate, that, under some circumstances, this is not true.

of temperature ceases when solidification begins; and if, as may occur under favorable conditions, a liquid is cooled below its melting-point, its temperature *rises* at once to the melting-point, when solidification begins. Heat, therefore, disappears when a body melts, and is generated when a liquid becomes solid.

It was seen (§ 90) that ice can be melted by friction; that is, by the expenditure of mechanical energy. Fusion is, therefore, work which requires the expenditure of some form of energy to accomplish it. Experiment shows that to melt a given mass of a substance always requires the same amount of energy. The heat required to melt unit mass of a substance is the *heat equivalent of fusion* of that substance. When a substance solidifies, it develops the same amount of heat as was required to melt it.

133. ***Nature of the Energy stored up in the Liquid.***— From the facts given above, as well as from the principles of the conservation of energy, it appears that the energy expended in melting a body is stored in the liquid. It is easy to see what must be the nature of this energy. When a body solidifies, its molecules assume certain positions in obedience to their mutual attractions. When it is melted, the molecules are forced into new positions in opposition to the attractive forces. They are, therefore, in positions of advantage with respect to these forces, and possess potential energy.

134. ***Determination of the Heat Equivalent of Fusion.***— The heat equivalent of fusion may be determined by the method of mixtures (§ 101), as follows: a mass P of the substance, ice, for example, at temperature t be-

low its melting-point, to insure dryness, is plunged into a mass P' of warm water at temperature T. When the ice is all melted, the resulting temperature is θ. If p be the water equivalent of the calorimeter, $(P+p)(T-\theta)$ is the heat given up by the calorimeter and its contents. Let c be the specific heat of ice, and x the heat equivalent of fusion. The ice absorbs, to raise its temperature to zero, Ptc calories; to melt it, Px calories; to warm the water after melting, $P\theta$ calories. We then have this equation,

$$Ptc + P\theta + Px = (P+p)(T-\theta),$$

from which x may be found.

Other calorimetric methods may be employed. The best experiments give, for the heat equivalent of fusion of ice, very nearly eighty calories.

CHAPTER IV.

GASES AND VAPORS.

135. ***The Gaseous State.*** — A gas may be defined as a highly compressible fluid. A given mass of gas has no definite volume. Its volume varies with every change in the external pressure to which it is exposed. A vapor is the gaseous state of a substance which at ordinary temperatures exists as a solid or a liquid.

136. ***Vaporization*** is the process of formation of vapor. There are two phases of the process, — *evaporation*, in which vapor is formed at the free surface of the liquid, and *ebullition*, in which the vapor is formed in bubbles in the mass of the liquid, or at the heated surface with which it is in contact.

137. ***Nature of the Process of Evaporation.*** — It has been seen (§ 91) that there are many reasons for believing that the molecules of solids and liquids are in a state of continual motion. It is not supposed that any one molecule maintains continuously the same condition of motion; but in the interaction of the molecules the motion of any one may be more or less violent, as it receives motion from, or gives up motion to, its neighbors. It can easily be supposed, that, at the exposed surface of the substance, the motion of a molecule may at times be so violent as to project it beyond the reach

of the mutual attractions. If this occur in the air or in a space filled with any gas, the molecule may be turned back, and made to rejoin the molecules in the liquid mass; but many will find their way to such a distance that they will not return. They then constitute a vapor of the substance. As the number of free molecules in the space above the liquid increases, it is plain that there may come a time when as many will rejoin the liquid as escape from it. The space is then *saturated* with the vapor. The more violent the motion in the liquid,—that is, the higher its temperature,—the more rapidly will the molecules escape, and the greater must be the number in the space above the liquid before the returning will equal the outgoing molecules. In other words, the higher the temperature, the more dense the vapor that saturates a given space. If the space above a liquid be a vacuum, the escaping molecules will at first meet with no obstruction, and, as a consequence, the space will be very quickly saturated with the vapor.

Experiment verifies all these deductions. Evaporation goes on continually from the free surfaces of many liquids, and even solids. It increases in rapidity as the temperature increases, and ceases when the vapor has reached a certain density, always the same for the same temperature, but greater for a higher temperature. It goes on very rapidly in a vacuum; but it is found that the final density of the vapor is no greater, or but little greater, than when some other gas is present. In other words, while a foreign gas impedes the motion of the outgoing molecules, and causes evaporation to go on slowly, it has very little influence upon the number of molecules that must be present in order that those that return may equal those that escape.

138. ***Pressure of Vapors.*** — As a liquid evaporates in a closed space, the vapor formed exerts a pressure upon the enclosure and upon the surface of the liquid, which increases as long as the quantity of vapor increases, and reaches a maximum when the space is saturated. This *maximum pressure* of a vapor increases with the temperature. When evaporation takes place in a space filled by another gas which has no action upon the vapor, the pressure of the vapor is added to that of the gas, and the pressure of the mixture is therefore the sum of the pressures of its constituents. The law was announced by Dalton, that the quantity of vapor which saturates a given space, and consequently the maximum pressure of that vapor, is the same whether the space be empty or contain a gas. Regnault has shown, that, for water, ether, and some other substances, the maximum pressure is slightly less when air is present.

139. ***Ebullition.*** — As the temperature of a liquid rises, the pressure which its vapor may exert increases, until a point is reached where the vapor is capable of forming, in the mass of the liquid, bubbles which can withstand the superincumbent pressure of the liquid and the atmosphere above it. These bubbles of vapor, escaping from the liquid, give rise to the phenomenon called *ebullition*, or boiling. Boiling may, therefore, be defined as the agitation of a liquid by its own vapor.

Generally speaking, for a given liquid ebullition always occurs at the same temperature for the same pressure; and, when once commenced, the temperature of the liquid no longer rises, no matter how intense the source of heat. This fixed temperature is called the boiling-point of the liquid. It differs for dif-

ferent liquids, and for the same liquid under different pressures. That the boiling-point must depend upon the pressure is evident from the explanation of the phenomenon of ebullition above given.

Substances in solution, if less volatile than the liquid, retard ebullition. While pure water boils at 100°, water saturated with common salt boils at 109°. The material of the containing vessel also influences the boiling-point. In a glass vessel the temperature of boiling water is higher than in one of metal. If water be deprived of air by long boiling, its temperature may be raised considerably above the boiling-point before ebullition commences. Under these conditions, the first bubbles of vapor will form with explosive violence. The air dissolved in water separates at a high temperature in minute bubbles. Into these the water evaporates, and, whenever the elastic force of the vapor is sufficient to overcome the superincumbent pressure, enlarges them, and causes the commotion that marks the phenomenon of ebullition. If no such openings in the mass of the fluid exist, the cohesion of the fluid, or its adhesion to the vessel, as well as the pressure, must be overcome by the vapor: hence the rise in temperature when air is absent.

140. ***Production of Vapor in a Limited Space.***—When a liquid is heated in a limited space the vapor generated accumulates, increasing the pressure, and causing the temperature to rise above the ordinary boiling-point. Cagniard-Latour experimented upon liquids in spaces but little larger than their own volumes. He found, that, at a certain temperature, the liquid suddenly disappeared; that is, it was converted into vapor in a space

but little larger than its own volume. It is supposed, that above the temperature at which this occurs, which is called the *critical temperature*, the substance cannot exist in the liquid state.

141. ***Spheroidal State.*** — If a liquid be introduced into a highly heated capsule, or poured upon a very hot plate, it does not wet the heated surface, but forms a flattened spheroid, which presents no appearance of boiling, but evaporates only very slowly. Boutigny has carefully studied these phenomena, and made known the following facts. The temperature of the spheroid is below the boiling-point of the liquid. The spheroid does not touch the heated plate, but is separated from it by a non-conducting layer of vapor. This accounts for the slowness of the evaporation. To maintain the spheroid, the temperature of the capsule must be much above the boiling-point of the liquid; for water it must be at least 200° C. If the capsule be allowed to cool, the temperature will soon fall below the limit necessary to maintain the spheroid, the liquid will moisten it, and there will be a rapid ebullition, with disengagement of a large amount of vapor. If a liquid of very low boiling-point, as liquid nitrous oxide, which boils at — 88°, be poured into a red-hot capsule, it will assume the spheroidal state; and, since its temperature cannot rise above its boiling-point, water, or even mercury, plunged into it will be frozen.

142. ***Liquefaction.*** — Only a certain amount of vapor can exist in a given space at a given temperature. If a space saturated with vapor be cooled, some of the vapor must condense into the liquid state. It is not necessary that the whole space be cooled; for, the vapor in the

cooled portion being condensed, its pressure is diminished, the vapor from the warmer portion flows in, to be in its turn condensed, and this continues till the whole is brought to the density and pressure due to the cooled portion. Any diminution of the space occupied by a saturated vapor will cause some of the vapor to become liquid, for, if it do not condense, its density and pressure must increase; but a saturated vapor is already at its maximum density and pressure.

If the vapor in a given space be not at its maximum density, its pressure will increase when its volume is diminished, until the maximum pressure is reached; when, if the temperature remain constant, further reduction of volume causes condensation into the liquid state, without further increase of density or pressure. This statement is true of several of the gases at ordinary temperatures. Chlorine, sulphurous acid, ammonia, nitrous oxide, carbonic acid, and several other gases, become liquid under sufficient pressure. Andrews has found, that, at a temperature of 30.92°, pressure ceases to liquefy carbonic acid. This is the critical temperature for that substance. The critical temperatures of oxygen, hydrogen, and the other so-called permanent gases, are so low that it is only by methods capable of yielding an extremely low temperature, combined with great pressure, that they can be liquefied. By the use of such methods any of the gases may be made to assume the liquid state. In the case of hydrogen, however, the low temperature necessary for its liquefaction has only been reached by allowing the gas to expand suddenly from a condition of great condensation, in which it has already been cooled to a very low point.

143. ***Pressure and Density of Non-saturated Gases and Vapors.*** — If a gas or vapor in the non-saturated condition be maintained at constant temperature, it follows very nearly Boyle's law (Mechanics, §§ 67 and 88). If its temperature be below its critical temperature, the product of volume by pressure diminishes, and near the point of maximum density the departure from the law may be considerable. At this point there is a sudden diminution of volume, the vapor assuming the liquid state. This liquid is at first much more compressible than ordinary liquids, but its compressibility diminishes with the pressure. The less the pressure and density in the gaseous state, the more nearly is Boyle's law fulfilled.

It has been seen already (§ 89) that gases expand as the temperature rises. Gay-Lussac was one of the first to measure accurately the expansion. He announced the following law, which bears his name: For each increment of temperature of 1° a gas expands by a constant fraction of its volume at zero. This is equivalent to saying that a gas has a constant co-efficient of expansion. Hence the following formulas.

Let V_0, V_t, represent the volumes at zero and t° respectively, and a the co-efficient of expansion. Then, the pressure remaining constant,

$$V_t = V_0(1 + at). \qquad (51)$$

If d_0, d_t, represent the densities at the two temperatures, we have, since densities are inversely as volumes,

$$d_t = \frac{d_0}{1 + at}. \qquad (52)$$

Charles discovered another very important law, —

that all gases expand equally for equal increments of temperature; in other words, the co-efficients of expansion of all gases are the same.

Later investigations, especially those of Regnault, show that these simple laws, like the law of Boyle, are not rigorously true, though they are very nearly so for all gases and vapors when not too near their points of maximum density. The common co-efficient of expansion is $a = 0.00367 = \frac{1}{273}$ very nearly.

From the law of Boyle we have, for a given mass of gas, temperature remaining constant,

$$V_p p = V_{p'} p' = \text{volume at pressure unity},$$

where V_p, $V_{p'}$, represent the volumes at pressure p and p' respectively.

From the law of Gay-Lussac we have, pressure remaining constant,

$$V_0 = \frac{V_t}{1 + at} = \frac{V_{t'}}{1 + at'}. \qquad (53)$$

If the temperature and pressure both vary, we have

$$\frac{V_{pt} p}{1 + at} = \frac{V_{p't'} p'}{1 + at'}; \qquad (54)$$

that is, if the volume of a given mass of gas be multiplied by the corresponding pressure and divided by the factor of expansion, the quotient is constant. The symbol V_{pt} may be read, volume at pressure p and temperature t.

144. ***Pressure of a Gas or Vapor at its Maximum Density.*** — It has been seen, that, for each gas or vapor at a temperature below the critical temperature, there is a maximum density, and consequently a maximum press-

ure which it can exert at that temperature. To each temperature, there corresponds a maximum pressure, which is higher as the temperature is higher. A gas in contact with its liquid in a closed space will exert its maximum pressure.

The relation between the temperature and the corresponding maximum pressure of a vapor is a very important one, and has been the subject of many investigations. The vapor of water has been especially studied, the most extensive and accurate experiments being those of Regnault.

Two distinct methods were employed, — one for temperatures below 50°, and the other for higher temperatures. The first consisted in observing the difference in height of two barometers placed side by side, the vacuum chamber of one containing a little water. The temperature was carried from zero to about 50°; both barometers being surrounded by the same medium, and in every way under the same conditions, except the presence of water and its vapor in one, and their absence from the other. The difference between the heights of the two, therefore, gave the pressure of the vapor at the temperature of the experiment.

The second method was founded on the principle that the vapor of a boiling liquid exerts a pressure equal to that of the atmosphere above it. The experiment consisted in boiling water in a closed space in which the air could be rarefied or condensed to a known pressure, and noting the temperature of the boiling liquid and the vapor above it. To prevent the accumulation of the vapor and the consequent change of pressure, a part of the space communicating with the boiler, was a tube

surrounded by a larger tube, forming an annular space, through which a stream of cold water was kept flowing. By this means the vapor was condensed as fast as formed, the water from its condensation flowing back into the boiler. By rarefying or compressing the air in the closed space, an artificial atmosphere of any desired pressure could be obtained, and maintained constant as long as was necessary for making the observations.

The temperature was determined by means of four thermometers placed in the boiler, two of them in the liquid and two in the vapor. The bulbs of the thermometers were in metal tubes, which protected them from the pressure, which, by compressing the bulb, would cause the thermometer to register too high a temperature.

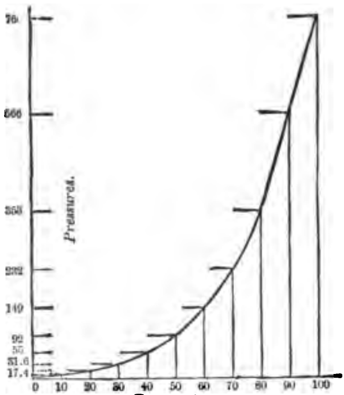

Fig. 51.

The results of Regnault's observations may be represented graphically, as in Fig. 51, where pressures are measured in the vertical, and temperatures in the horizontal, direction. It is seen that the pressure varies very rapidly with the temperature.

145. ***Kinetic Theory of Gases.*** — According to the kinetic theory, a perfect gas consists of an assemblage of *free, perfectly elastic* molecules in constant motion. Each molecule moves in a straight line with a constant velocity, until it encounters some other molecule, or the

sides of the vessel. The impact of the molecules upon the sides of the vessel are so numerous that their effect is a continuous constant force or *pressure.*

The entire independence of the molecules follows from the fact, that, when gases or vapors are mixed, the pressure of one is added to that of the other; that is, the pressure of the mixture is the sum of the pressures of the separate gases. It follows from this independence that no energy is required to separate the molecules; in other words, no internal work needs be done to expand a gas. This was demonstrated experimentally by Joule, who showed, that, when a gas expands without performing external work, it is not cooled.

Let us consider the laws and phenomena of gases from the standpoint of the kinetic theory. The action between two molecules, or between a molecule and solid wall, must be of such a nature that no energy is lost; that is, the sum of the kinetic energies of all the molecules must remain constant. Whatever be the nature of this action, it is evident, that, when a molecule strikes a solid stationary wall, it must be reflected back with a velocity equal to that before impact. If the velocity be resolved into two components, one parallel and the other normal to the wall, the parallel component remains unchanged, while the normal component is changed from $+v$ before impact, to $-v$ after impact. The change of velocity is, therefore, $2v$; and if θ be the duration of impact, the mean acceleration is $\frac{2v}{\theta}$, and the mean force of impact $p = m\frac{2v}{\theta}$, where m is the mass of the molecule: hence

$$p\theta = 2mv; \qquad (55)$$

that is, the product of the mean force by the duration of impact equals the change of momentum of the particle, and the total force or pressure exerted upon a given area, multiplied by a time, will equal the sum of the changes of momentum of the molecules impinging upon that area during that time: for the number of molecules, even in the smallest mass of gas upon which we can experiment, is so great that the impacts will be continuous, and the same at any one instant as at any other. The number of impacts occurring in unit time may be determined from the following considerations: Suppose the molecules between two parallel walls at distance s from each other: any molecule may be supposed to suffer reflection from one wall, pass across to the other, be reflected back to the first, and so on. Whatever may be the effect of the mutual collisions of the molecules, the number of impacts upon the surface considered will be the same as though each one preserved its rectilinear motion unchanged, except when reflected from the solid walls. The time required for a molecule to pass across the space between the two walls and back is, if v be the component of the velocity normal to the surface, $\frac{2s}{v}$; and the number of impacts upon the first surface in unit time is $\frac{v}{2s}$. The sum of the changes of the momentum of this molecule is

$$2mv \times \frac{v}{2s} = \frac{mv^2}{s}.$$

The total pressure P upon unit area, multiplied by the time, one second, equals the sum of the changes

of momentum of all the molecules in the space; that is,

$$P \times 1 = \Sigma\frac{mv^2}{s}.$$

The sum of $\frac{mv^2}{s}$ is proportional to the number, n', of molecules in the space. These molecules must be considered as moving in all directions and with various velocities. But the velocity of any molecule may be resolved in the direction of three rectangular axes, one normal to the surface and the other two parallel to it; and, since the number of molecules in any finite volume of gas is practically infinite, the effect upon the wall due to their real motions would be the same as would result from a motion of one-third the molecules in each of the three directions with the mean velocity. Now, the molecules moving in the two directions parallel to the wall have no influence upon the pressure. We have, then, the pressure upon unit surface, $P = \frac{1}{3}\frac{n'}{s}mv^2$.

But $\frac{n'}{s}$ = the number of molecules in unit volume $= n$: hence

$$P = \tfrac{1}{3}nmv^2, \tag{56}$$

and the pressure upon unit area is equal to one-third the number of molecules in unit volume at that pressure multiplied by twice the kinetic energy of each molecule.

Suppose, now, the volume of the gas be changed to V, without change of temperature. The number of molecules in unit volume is now $\frac{n}{V}$, and the pressure

$P_1 = \frac{1}{3}\frac{n}{V}mv^2$, whence $P_1 V = \frac{1}{3}nmv^2$, equals a constant, since n and m are constant for the same mass of gas, and v is constant if there is no change of temperature. But PV equal to a constant is Boyle's law.

From the law of Gay-Lussac we have, if P = pressure at $t°$, and P_0 = pressure at zero,

$$P = P_0(1 + at).$$

$a = 0.00367 = \frac{1}{273}$ very nearly: hence

$$P = P_0\left(1 + \frac{t}{273}\right). \qquad (57)$$

If $t = -273°$,

$$P = P_0\left(1 - \frac{273}{273}\right) = 0;$$

that is, at 273° below zero Centigrade the pressure is null; and since $P = \frac{1}{3}nmv^2$, it follows, that, at this temperature, $v = 0$, or the molecules are at rest. This is therefore called the *absolute zero.*

In studying the expansion of gases, it is very convenient to use a scale of temperatures whose zero-point is at the absolute zero. Temperatures reckoned upon this scale are called absolute temperatures. Let T represent a temperature upon the absolute scale: then $T = t + 273$, and formula (57) becomes

$$P = P_0\frac{T}{273}.$$

Substituting the value of P from (56),

$$\frac{1}{3}nmv^2 = P_0\frac{T}{273},$$

$$T = \frac{1}{3}n\frac{273}{P_0}mv^2.$$

That is, the absolute temperature of a gas is proportional to the kinetic energy of the molecules.

It has been seen already (§ 90), that, when a gas is compressed, a certain amount of heat is generated.

Suppose a cylinder with a tight-fitting piston. As long as the piston is at rest, each molecule that strikes it is reflected with a velocity equal to that before impact: but if the piston be forced into the cylinder, each molecule, as it is reflected, has its velocity increased; and, as was shown above, this is equivalent to a rise in temperature. It can be shown that the increase of kinetic energy in this case is precisely equal to the work done in forcing the piston into the cylinder against the pressure of the gas. On the other hand, if the piston be pushed backward by the force of the impact of the molecules, there will be a loss of velocity by reflection from the moving-surface, kinetic energy equal in amount to the work done upon the piston disappears, and the temperature falls.

146. ***Mean Velocity of Molecules.*** — Equation (56) enables us to determine the mean velocity of the molecules of a gas whose density and pressure are known, since nm is the mass of the gas in unit volume.

Solving the equation with reference to v, and substituting the known values for hydrogen, namely, $P = 1013373$ dynes per square centimetre, and nm, or density, $=$ 0.00008954 grams per cubic centimetre, we have 184260 centimetres per second, or a little more than one mile per second.

147. ***Elasticity of Gases.*** — It has been shown (Mechanics, § 67) that the modulus of elasticity of a gas, obeying Boyle's law, is numerically equal to the pressure. This

is the elasticity for constant temperature. But, as was seen (§ 145), when a gas is compressed, it is heated; and heating a gas increases its pressure. Under ordinary conditions, therefore, the ratio of a small increase of pressure to the corresponding decrease of the unit volume (= elasticity) is greater than when temperature is constant. It is important to consider the case when all the heat generated by the compression is retained by the gas. The elasticity is then a maximum, and is called the elasticity when no heat is allowed to enter or escape.

Let *mn* (Fig. 52) be a curve representing the relation between volume and pressure for constant temperature, x representing volume and y pressure. Such a curve is called an *isothermal* line. It is plain that to each temperature must correspond its own isothermal line. If, now, we suppose the gas to be compressed, and no heat to escape, it is plain, that, if the volume diminish from OC to OG, the pressure will become greater than GD; say, GM. If a number of such points as M be found, and a line be drawn through them, it represents the relation between volume and pressure when no heat enters or escapes. It is called an *adiabatic line.* It evidently makes a greater angle with the horizontal than the isothermal.

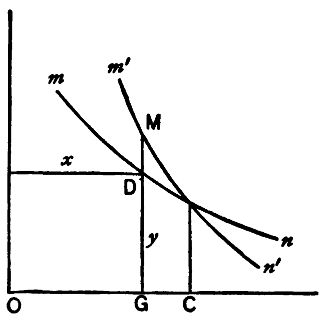

Fig. 52.

148. ***Specific Heats of Gases.*** — In § 145 it is seen that

the temperature of a gas is proportional to the kinetic energy of the molecules. To warm a gas without change of volume is, therefore, only to add to this kinetic energy. If, however, the gas be allowed to expand when heated, the molecules lose energy by impact upon the receding surface; and this, together with the kinetic energy due to the rise in temperature, must be supplied from the source of heat. It has been seen that the loss of energy resulting from impact upon a receding surface is equal to the work done by the gas in expanding.

The amount of heat necessary to raise the temperature of unit mass of a gas one degree, while the volume remains unchanged, is called *the specific heat of the gas at constant volume.* The amount of heat necessary to raise the temperature of unit mass of a gas one degree when expansion takes place without change of pressure, is called the specific heat of the gas *at constant pressure.*

From what has been said above, it is evident that the difference between these two quantities of heat is the equivalent of the work done by the expanding gas.

The determination of the relation of these two quantities is a very important problem.

The specific heat of a gas at constant pressure may be found by passing a current of warmed gas through a calorimeter, — method of mixtures. There are great difficulties in the way of an accurate determination, because of the small density of the gas, and the time required to pass enough of it through the calorimeter to obtain a reasonable rise of temperature. The various sources of error produce effects which are sometimes as great as, or even greater than, the quantity to be measured. It is beyond the scope of this work to

describe in detail the means by which the effects of the disturbing causes have been determined or eliminated. The specific heats of a few gases are given below.

Air	0.23741
Oxygen	0.21851
Hydrogen	3.4090
Carbonic acid	0.2169

The specific heat of a gas at constant volume is generally determined from the ratio between it and the specific heat at constant pressure. The first determination of this ratio was accomplished by Clement and Desormes.

The theory of the experiment may be understood from the following considerations: Let a unit mass of gas at any temperature t and volume V be raised in temperature 1° at constant volume. It absorbs heat, which we will represent by c, the specific heat at constant volume. Now let the gas expand against a mean pressure H, maintaining its temperature at 1° until it returns to its original pressure. It absorbs an additional quantity of heat equivalent to the work done in expanding, $= H \times$ the increase in volume. The increase in volume is the same as would have occurred had the gas been raised in temperature 1°, and allowed to expand from the beginning; that is, $\frac{\alpha V}{1 + \alpha t} = \alpha V_0$, where $V_0 =$ volume at zero. The work done is $H\alpha V_0$, and the heat equivalent is $\frac{H\alpha V_0}{J}$, where J has its usual signification. The heat absorbed is then

$$C = c + \frac{H\alpha V_0}{J}, \tag{58}$$

which is the specific heat at constant pressure. Hence

$$\frac{C}{c} = 1 + \frac{1}{c}\frac{H\alpha V_0}{J}. \qquad (59)$$

If, in the case considered above, the gas had expanded without receiving any heat, the work $H\alpha V_0$ would have been done at the expense of its own internal energy, and the temperature would have fallen. The performance of this work is equivalent to abstracting the quantity of heat, $\frac{H\alpha V_0}{J}$, which would lower the temperature $\frac{1}{c}\frac{H\alpha V_0}{J}$ degrees, since a quantity c of heat raises the temperature one degree. Represent this change of temperature by θ. It corresponds to a change in the unit volume of $\frac{\alpha}{1+\alpha t}$. Substituting θ for $\frac{1}{c}\frac{H\alpha V_0}{J}$ in equation (59), we have $\frac{C}{c} = 1 + \theta$. It is the object of the experiment to find θ. The method of Clement and Desormes is as follows: —

A large flask is furnished with a stop-cock having a large opening, and a very sensitive manometer which shows the difference between the pressure in the flask and the pressure of the air. The air in the flask is first rarefied, and left to assume the temperature of the surrounding atmosphere. Suppose the pressure now to be $H - h$, H being the height of the barometer, and h the difference between the pressure in the flask and the pressure of the atmosphere, as shown by the manometer. The large stop-cock is then suddenly opened for a very short time only; the air rushes in,

re-establishing the atmospheric pressure, compressing the gas, and raising its temperature. The volume of the air becomes $1 - \phi$, where the original volume is taken as unity, and ϕ represents its reduction; and, if there were no change of temperature, the pressure would be $\frac{H-h}{1-\phi}$. If the temperature increase θ'°, and become $t + \theta'$, the pressure will be

$$\frac{H-h}{1-\phi} \times \frac{1 + a(t+\theta')}{1 + at} = H, \qquad (60)$$

the atmospheric pressure.

The flask is now left till it returns to the temperature of the atmosphere, t, when the manometer shows a fall of pressure h', and we have

$$\frac{H-h}{1-\phi} = H - h'. \qquad (61)$$

From these two equations we have

$$\phi = \frac{h-h'}{H-h'}, \quad \theta' = \frac{(1 + at)h'}{a(H-h')}.$$

Suppose, now, the change of volume had been $\frac{a}{1+at}$, then the change of temperature would have been θ; and, since change of volume is proportional to change of temperature,

$$\phi : \frac{a}{1+at} = \theta' : \theta;$$

hence

$$\theta = \frac{\theta' \frac{a}{1+at}}{\phi};$$

or, substituting the values of ϕ and θ', we have

$$\theta = \frac{h'}{H-h'} \times \frac{H-h'}{h-h'} = \frac{h'}{h-h'}.$$

Now,

$$\frac{C}{c} = 1 + \theta,$$

$$\frac{C}{c} = 1 + \frac{h'}{h - h'} = \frac{h}{h - h'}. \qquad (62)$$

It is evident that the specific heat may be computed from formula (58).

149. ***The Two Specific Heats of a Gas have the Same Ratio as the Two Elasticities.*** — Suppose a gas whose mass is unity and volume V to rise in temperature at constant pressure from $t°$ to $(t + \Delta t)°$, Δt representing a very small increment of temperature. The heat consumed will be $C\Delta t$, and the increase of volume $\psi\Delta t$, where ψ represents $V\alpha$. Now, suppose the volume to remain constant, the amount of heat required to cause the rise of temperature Δt will be $c\Delta t$: hence the amount of heat, $C\Delta t$, will cause a rise of temperature $\frac{C}{c}\Delta t$; and the same rise of temperature will occur if the gas, after first being allowed to expand, be compressed to its initial volume. Such a compression would be attended by an increase of pressure, which we will call Δp. The ratio between this and the corresponding change of volume is

$$\frac{\Delta p}{\psi\Delta t} = E_h, \qquad (63)$$

where E_h is the elasticity under the condition that no heat enters or escapes.

If, now, the heat produced by compression be allowed to escape, there will remain the quantity $c\Delta t$, and the increment of pressure will be reduced to $\delta p = \Delta p\frac{c}{C}$.

This is the increase of pressure that will occur if the gas be compressed by the amount $\psi\Delta t$ without change of temperature: hence

$$\frac{\delta p}{\psi\Delta t} = E_t, \tag{64}$$

where E_t is the elasticity for constant temperature. Dividing (63) by (64), we have

$$\frac{E_k}{E_t} = \frac{\frac{\Delta p}{\psi\Delta t}}{\frac{\delta p}{\psi\Delta t}} = \frac{\Delta p}{\delta p} = \frac{\Delta p}{\Delta p\frac{c}{C}} = \frac{C}{c};$$

that is, the two elasticities have the same ratio as the two specific heats of a gas.

In acoustics it will be seen that the velocity of sound in any medium is equal to the square root of the quotient of the elasticity divided by the density of the medium; that is,

$$velocity = \sqrt{\frac{E}{D}}. \tag{65}$$

It will be seen also, that, in the progress of a sound-wave, the air is alternately compressed and rarefied, the compressions and rarefactions occurring in such rapid succession that there is no time for any transfer of heat. If the above equation, (65), be applied to air, the E becomes E_k, or the elasticity under the condition that no heat enters or escapes. Since we know the density of the air and the velocity of sound, E_k can be computed. In "Mechanics," § 67, it is shown that E_t is numerically equal to the pressure: hence we have the values of the two elasticities of air, and, as seen above, their ratio is the ratio of the two specific heats of air.

150. ***Examples of Energy absorbed by Vaporization.*** — When a liquid boils, its temperature remains constant, however intense the source of heat. This shows that the heat applied to it is consumed in producing the change of state. Evaporation absorbs heat. By promoting evaporation, intense cold may be produced. In a vacuum, water may be frozen by its own evaporation. If a liquid is heated to a temperature above its ordinary boiling-point under pressure, relief of the pressure is followed by a very rapid evolution of vapor and a rapid cooling of the liquid. Liquid nitrous oxide at a temperature of zero is still far above its boiling-point, and its vapor exerts a pressure of about thirty atmospheres. If the liquid be drawn off into an open vessel, it at first boils with extreme violence, but is soon cooled to its boiling-point for the atmospheric pressure, about — 88°, and then boils away slowly, while its temperature remains at that low point.

151. ***Heat Equivalent of Vaporization.*** — It is plain, from what has preceded (§ 137), that the formation of vapor is work requiring the expenditure of energy for its accomplishment. Each molecule that is shot off into space obtains the motion which projected it beyond the reach of the molecular attraction, at the expense of the energy of the molecules that remain behind. A quantity of heat disappears when a liquid evaporates; and experiment demonstrates, that to evaporate a kilogram of a liquid at a given temperature always requires the same amount of heat. This is the *heat equivalent of vaporization.* When a vapor condenses into the liquid state, heat is generated the same in amount as disappears when the liquid assumes the state of vapor. The heat

equivalent of vaporization is determined by passing the vapor at a known temperature into a calorimeter, there condensing it into the liquid state, and noting the rise of temperature in the calorimeter. This, it will be seen, is essentially the method of mixtures. Many experimenters have given attention to this determination; but here, again, the best experiments are those of Regnault. He determined what he called the *total heat of steam* at various pressures. By this was meant the heat required to raise the temperature of a kilogram of water from zero to the temperature of saturated vapor at the pressure chosen, and then convert it wholly into steam. The results of his experiments give, for the heat equivalent of vaporization of water at 100°, 537 calories. That is, he found, that by condensing a kilogram of steam at 100° into water, and then cooling the water to zero, 637 calories were obtained. But almost exactly 100 calories are derived from the water cooling from 100° to zero: hence $637 - 100 = 537$ calories is the heat equivalent of vaporization at 100°.

152. ***Dissociation.*** — It has already been noted (§ 89), that, at high temperatures, compounds are separated into their elements. To effect this separation, the powerful forces of chemical affinity must be overcome, and a considerable amount of energy must be consumed.

153. ***Heat Equivalent of Dissociation and Chemical Union.*** — From the principle of the conservation of energy, it would be supposed that the energy required for dissociation is the same as that developed by the re-union of the elements. The heat equivalent of chemical union is not easy to determine, because the process is usually complicated by changes of physical state.

We may cause the union of carbon and oxygen in a calorimeter, and, bringing the products of combustion to the temperature of the elements before the union, measure the heat given to the instrument; but the carbon has changed its state from a solid to a gas, and some of the chemical energy must have been consumed in that process. The heat measured is the *available* heat. Several experimenters have made determinations of the available heat of chemical union. Among these, Andrews, and Favre and Silberman have performed experiments with great care, and obtained results, a few of which are given in the following table, which gives the number of calories produced by the combustion of one gram of the substance in oxygen: —

Hydrogen	34462
Carbon (wood charcoal)	8080
Native sulphur	2261

CHAPTER V.

HYGROMETRY.

154. ***Object of Hygrometry.*** — Hygrometry has for its object the determination of the state of the air with regard to moisture.

155. ***Amount of Vapor in a Given Volume of Air.*** — This may be determined directly by passing a known volume of air through tubes containing some substance which will absorb the moisture, and determining the increased weight of the tubes and their contents. The quantity of vapor in grams contained in a cubic metre of air is called its absolute humidity. Methods of determining this quantity indirectly will be given below.

156. ***Pressure of the Vapor.*** — It has been seen (§ 138), that, when two or more gases occupy the same space, each exerts its own pressure independently of the others. If we can determine this pressure, it is easy to compute the quantity of moisture in the air.

It appears, also, that the pressure exerted by the vapor in the air would at a certain temperature be its maximum pressure. Now, if any small portion of the space be cooled till its temperature is below that at which the pressure exerted is the maximum pressure, a portion of the vapor will condense into the liquid form. If, then, we determine the temperature at which

condensation begins, the maximum pressure of the vapor for this temperature, which may be found from tables, is the real pressure of the vapor in the air. The mass of vapor in a metre cube of air may then be computed as follows: A metre cube contains 1293.2 grams of air at zero and pressure 760 millimetres. At the pressure p of the vapor, and temperature t of the air at the time of the experiment, the same space would contain

$$1293.2 \times \frac{p}{760} \times \frac{1}{1 + at}$$

grams of air; and, since the specific gravity of vapor of water in relation to air is 0.623, a metre cube would contain

$$1293.2 \times \frac{p}{760} \times \frac{1}{1 + at} \times 0.623 \qquad (66)$$

grams of vapor.

157. ***Dew Point.*** — The temperature at which the vapor of the air begins to condense is called the *dew point.* It is determined by means of instruments called dew-point hygrometers, which are instruments so constructed that a small surface exposed to the air may be cooled till moisture deposits upon it, when its temperature is accurately determined. The Alluard hygrometer consists of a metal box about one and a half centimetres square and four centimetres deep. Two tubes pass through the top of the box, — one terminating just inside, and the other extending to the bottom. One side of the box is gilded and polished, and is so placed that the gilded surface lies on the same plane with, and in close proximity to, a gilded metal plate. The box is

partly filled with ether, and the short tube connected with an inspirator. Air is thus drawn through the longer tube, and, bubbling up through the ether, causes rapid evaporation, which soon cools the box, and causes a deposit of dew upon the gilded surface. The presence of the gilded plate forming an extension of the surface of the box helps very much in recognizing the beginning of the deposit of dew, by the contrast between it and the dew-covered surface of the box. A thermometer plunged in the ether gives its temperature, and another outside gives the temperature of the air.

158. ***Relative Humidity.*** — The amount of moisture that the air may contain depends upon its temperature. The dampness or dryness of the air does not depend upon the absolute amount of moisture it contains, but upon the ratio of this to the amount it might contain if saturated. The *relative humidity* is the ratio of the amount of moisture in the air to that which would be required to saturate it at the existing temperature. Since non-saturated vapors follow nearly Boyle's law, this ratio will be very nearly the ratio of the actual pressure to the possible pressure for the temperature. Both these pressures may be taken from the tables. One corresponds to the dew point, and the other to the temperature of the air.

CHAPTER VI.

THERMO-DYNAMICS.

159. ***First Law of Thermo-dynamics.*** — This law may be thus stated: Where heat is transformed into work, or work into heat, the quantity of work is mechanically equivalent to the quantity of heat. The experiments of Joule and Rowland establishing this law, and determining the mechanical equivalent, have already been described (§ 104).

160. ***Second Law of Thermo-dynamics.*** — When heat is converted into work under the conditions that exist on the earth's surface, only a comparatively small proportion of the heat drawn from the source can be transformed: the remainder is given up to a refrigerator, which in some form must be an adjunct of every heat-engine, and still exists as heat, which, so far as any useful effect is concerned, is wasted. It can be shown that the heat which can be converted into work, bears to that which must be drawn from the source of heat a certain simple ratio depending upon the temperatures of the source and refrigerator. The second law of thermo-dynamics asserts this relation. The ratio between the heat converted into work and that drawn from the source is called the efficiency of the engine.

To convert heat into mechanical work, it is neces-

sary that the heat should act through some substance called the working substance; for instance, steam in the steam-engine, air in the hot-air engine. In studying the transformation of heat into work, it is an essential condition, which must not be lost sight of, that the working substance must, after passing through a cycle of operations, return to the same condition as at the beginning; for, if the substance is not in the same condition at the end as at the beginning, internal work may have been done, or internal energy expended, which would increase or diminish the work apparently developed from the heat.

To determine the laws governing the conversion of heat into work, recourse is had to a conception of Carnot, — that of an engine completely reversible in all its mechanical and physical operations. Such an engine must, when running forward, receive from a source a certain quantity of heat H, transfer to a refrigerator a certain quantity of heat h, and perform a certain amount W of mechanical work. If it be perfectly reversible, it will, by the performance upon it of the amount of work W, take from the refrigerator the quantity of heat h, and restore to the source the amount H. Such an engine will convert into work, under given conditions, as large as possible a proportion of the heat taken from the source. For, let there be two engines, A and B, of which B is reversible, working between the same source and refrigerator. If possible let A perform more work than B, while taking from the source the same amount of heat. If W be the work it performs, and w the work B performs, B will, from its reversibility, by the performance upon it of the work w (less than W),

restore to the source the amount of heat, H, which it takes away when running forward. Let A be employed to run B backward: A will take from the source a quantity of heat, H, and perform work, W. B will restore the heat H to the source by the performance upon it of work, w. The system will then continue running, developing the work $W - w$; while the source loses no heat. It must be, then, that A gives up to the refrigerator less heat than B takes away; and the refrigerator must be growing colder. For the purposes of this discussion, we may assume that all surrounding bodies, except the refrigerator, are at the same temperature as the source: hence the work $W - w$, performed by the system of two engines, must be performed by means of heat taken from a body colder than all surrounding bodies. Now, this is contrary to all experience. Heat cannot be drawn from a cold body, and made to do work. The hypothesis with which we started must, therefore, be false; and we must admit that no engine, no contrivance for converting heat into work, can under similar conditions, and while taking the same heat from the source, perform more work than a reversible engine. It follows that all reversible engines, *whatever the working substance*, have the same efficiency. This is a most important conclusion. In view of it, we may, in studying the conversion of heat into work, choose for the working substance the one which presents the greatest advantage for the study. Of all substances, the properties of gases are best known. These, then, are best adapted to this purpose. Assuming a perfect gas as the working substance, let us see how the results sought may be reached. Carnot employed a cycle of four

operations, perfectly reversible, returning the substance at the end, to the same condition as at the beginning. The effect of every operation of this cycle, when applied to a gas, may be completely predicted.

Let the gas be enclosed in a cylinder having a tightly fitting piston. Suppose the cycle to begin by a depression of the piston, compressing the gas, without loss or gain of heat, till the temperature rises from θ to t; t being the temperature of the source, and θ that of the refrigerator. In Fig. 53, let Oa represent the volume, and Aa the pressure at the beginning. If the gas be compressed till its volume becomes Ob, its pressure will be bB, where AB is an adiabatic line. This is the *first operation*. For the *second operation*, let the piston rise, increasing the volume from b to c at the constant temperature of the source. The pressure will fall from bB to cC, BC being the isotherm for temperature t. During this operation, heat H must be taken from the source, to maintain the constant temperature t. For the *third operation*, let the piston still ascend, increasing the volume from c to d without loss or gain of heat, till the temperature falls from t to θ, the temperature at which the cycle began. CD, representing the fall of pressure, is an adiabatic line. For the *fourth operation*, let the piston be depressed to the starting-point, and the gas maintained at the constant temperature θ of the refrigerator. The volume becomes Oa, and the pressure aA, as at the beginning. Now let us consider the work

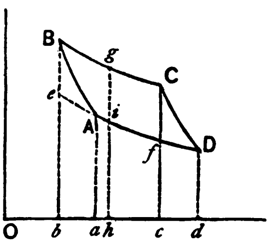

Fig. 53.

done in each operation. While the piston is being depressed through the volume represented by *ab*, work must be performed upon it equal to *ab* × the mean pressure exerted upon the piston. This mean pressure lies between *Aa* and *Bb*, and the product of this by *ab* is evidently the area *ABba*. In the same way it is shown, that, when the gas expands from *b* to *c*, it performs work represented by the area *BCcb;* and again, in the third operation, it performs work represented by *CDdc*. In the fourth operation, when the gas is compressed, work must be done upon it represented by the area *ADda*. During the cycle, therefore, work is done by the gas represented by *BCDdb*, and work is done *upon* the gas represented by *BADdb*. The difference represented by the area *ABCD* is the work done by the engine in one revolution. Since the gas is in all respects in the same condition at the end as at the beginning of the cycle, no work could have been developed from it; and the work which the engine does must have been derived from the heat communicated to the gas during the second operation.

Now, it has been shown, that, when a gas expands, no internal work is done in separating the molecules, and when it expands at constant temperature, no change occurs in the internal kinetic energy: the heat which is imparted to the gas during the second operation is, therefore, the equivalent to the work done by the gas upon the piston, and may be represented by the area *BCcb*. It will be seen, also, that the heat given up to the refrigerator during the fourth operation is represented by the area *ADda*, and that heat, the equivalent of the work performed by the engine, represented by the

area $ABCD$, has disappeared. Of the heat withdrawn from the source, then, only the fraction $\frac{area\ ABCD}{area\ BCcb}$ is converted into work. This fraction is the efficiency of the engine.

Now let the operations of the cycle be reversed. Starting from a, the gas expands at temperature θ, absorbs heat h the same as it gave up when compressed, and performs work represented by $ADda$; next, it is compressed, without loss of heat, till its temperature rises to t, and work represented by $DCcd$ is done upon it; next, it is still further compressed at temperature t, till its volume becomes Ob, and its pressure, Bb. During this operation, it gives up the heat H which it absorbed during the direct action, and work represented by $CBbc$ is done upon it. Lastly, it expands to the starting-point, and falls to its initial temperature. It will be seen, that each operation is the reverse in all respects of the corresponding operation of the direct action, and that, during the cycle, work represented by the area $ABCD$ must be performed *upon* the engine; while heat h is taken from the refrigerator, and heat H is transferred to the source. Such an engine is, therefore, a reversible engine; and it converts into work as large a proportion of the heat derived from the source as is possible under the circumstances. An inspection of the figure shows, that, since the line BC remains constant as long as the amount of heat H and the temperature t of the source remain constant, the only way to increase the proportion of work derived from a given amount of heat H is to increase the difference of temperature between the source and the refrigerator; that is, to increase the

area $ABCD$, the line AD must be taken lower down. The proportion of heat which can be converted into work depends, therefore, upon the difference of temperature between source and refrigerator. To determine the nature of this dependence, suppose the range of temperature so small that the sides of the figure $ABCD$ may be considered straight and parallel. Produce AD to e, and draw gh representing the mean pressure for the second operation. Now $ABCD = eBCf = Be \times bc = gi \times bc$. Also $BCcb = gh \times bc$. Then

$$\frac{\text{area } ABCD}{\text{area } BCcb} = \frac{gi \times bc}{gh \times bc} = \frac{gi}{gh} = \frac{H - h}{H}.$$

But gh is the pressure corresponding to volume Oh and temperature t, hi is the pressure corresponding to the same volume and temperature θ. These pressures are proportional to the absolute temperatures (§ 145); that is, if t and θ are temperatures on the absolute scale,

$$\frac{ih}{gh} = \frac{\theta}{t}$$

and

$$\frac{gi}{gh} = \frac{H - h}{H} = \frac{t - \theta}{t}: \qquad (67)$$

hence

$$\textit{efficiency} = \frac{\text{area } ABCD}{\text{area } BCcb} = \frac{t - \theta}{t}. \qquad (68)$$

This proportion has been derived upon the supposition that the range of temperature was very small: but it is equally true for any range; for, let there be a series of engines of small range, of which the second has for a source the refrigerator of the first, the third has for

a source the refrigerator of the second, and so on. The first takes from the source the heat H, and gives to the refrigerator the heat h, working between the temperatures t and θ. The second takes the heat h from the refrigerator of the first, and gives to its own refrigerator the heat h_1, working between the temperatures θ and θ_1, and similarly for the others; then, from (67),

$$\frac{H-h}{H}=\frac{t-\theta}{t},$$
$$\frac{h-h_1}{h}=\frac{\theta-\theta_1}{\theta},$$
$$\frac{h_1-h_2}{h_1}=\frac{\theta_1-\theta_2}{\theta_1},$$
$$\frac{h_{n-1}-h_n}{h_{n-1}}=\frac{\theta_{n-1}-\theta_n}{\theta_{n-1}},$$

or

$$\frac{h}{H}=\frac{\theta}{t},$$
$$\frac{h_1}{h}=\frac{\theta_1}{\theta},$$
$$\frac{h_2}{h_1}=\frac{\theta_2}{\theta_1},$$
$$\frac{h_n}{h_{n-1}}=\frac{\theta_n}{\theta_{n-1}};$$

multiplying,

$$\frac{h}{H}\times\frac{h_1}{h}\times\frac{h_2}{h_1}\times\cdots\frac{h_n}{h_{n-1}}=\frac{\theta}{t}\times\frac{\theta_1}{\theta}\times\frac{\theta_2}{\theta_1}\times\cdots\frac{\theta_n}{\theta_{n-1}},$$

or

$$\frac{h_n}{H}=\frac{\theta_n}{t}$$

and

$$\frac{H-h_n}{H}=\frac{t-\theta_n}{t}.$$

Hence it appears, that, in a perfect heat-engine, the heat converted into work is to the heat received, as the difference of temperature between the source and the refrigerator is to the absolute temperature of the source. This ratio can become unity only when $\theta = 0°$, or when the refrigerator is at the absolute zero of temperature. Since the difference of temperatures between which it is practicable to work is always small compared to the absolute temperature of the source, a perfect heat-engine can convert into work only a small fraction of the heat it receives.

161. ***Absolute Scale of Temperatures.*** — An absolute scale of temperatures, formed upon the assumed properties of a perfect gas, has already been described (§ 145). No such substance as a perfect gas exists; but, since (§ 160) any two temperatures on the absolute scale are to each other as the heat taken from the source at the higher temperature is to the heat transferred to the refrigerator at the lower, by a perfect engine, any substance whose properties we know with sufficient exactness to draw its isothermal and adiabatic lines, may be used as a thermometric substance, and, by means of it, an absolute scale of temperatures may be constructed. For, in Fig. 54 let BB' be an isothermal line, corresponding to temperature t, say, of boiling water at a standard pressure. Let $\beta\beta'$ be the isotherm for the temperature t_0 of melting ice, and let bb' be an isotherm for an intermediate temperature. Let $B\beta$, $B'\beta'$, be adiabatic lines, such that, if the substance expand at constant temperature t from the condition B to the condition B', the equivalent in heat of one mechanical unit of energy will be absorbed. If, now, the figure $BB'\beta'\beta$ represent Car-

not's cycle, the heat given to the refrigerator at temperature t_0, measured in mechanical units, is less than the heat taken from the source at temperature t, by the

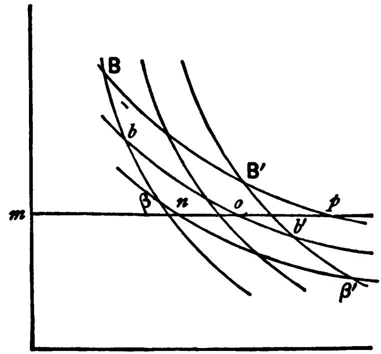

Fig. 54.

energy represented by the area $BB'\beta'\beta$, or, the heat given to the refrigerator $= 1 -$ area $B\beta'$: hence

$$\frac{t}{t_0} = \frac{1}{1 - area\ B\beta'},$$

and

$$\frac{t - t_0}{t} = \frac{area\ B\beta'}{1}.$$

Now, if $t - t_0 = 100°$, as in the Centigrade scale, we have

$$\frac{100}{t} = \frac{area\ B\beta'}{1},$$

$$t = \frac{100}{area\ B\beta'}. \qquad (69)$$

If θ be the temperature corresponding to the isotherm bb', we have, as above,

$$\frac{t}{\theta} = \frac{1}{1 - area\, Bb'},$$

$$\theta = t(1 - area\, Bb') = \frac{100}{area\, B\beta'}(1 - area\, Bb'). \quad (70)$$

If, now, it be proposed to use the substance as a thermometric substance by noting its expansion at constant pressure, take *Om* to represent that pressure, and draw the horizontal line *mnop*; *mn* is the volume of the substance at temperature t_0, *mo* the volume at temperature θ, and *mp* the volume at temperature *t*.

This method of constructing an absolute scale of temperature was proposed by Sir William Thomson.

162. ***The Steam-Engine.*** — The steam-engine in its usual form consists essentially of a piston, moving in a closed cylinder, which is provided with passages and valves by which steam can be admitted and allowed to escape. A boiler heated by a suitable furnace supplies the steam. The valves of the cylinder are opened and closed automatically, admitting and discharging the steam at the proper times to impart to the piston a reciprocating motion, which may be converted into a circular motion by means of suitable mechanism.

There are two classes of steam-engines, — condensing and non-condensing. In condensing engines the steam, after doing its work in the cylinder, escapes into a condenser, kept cold by a circulation of cold water. Here the steam is condensed into water; and this water, with air or other contents of the condenser, is removed by an "air-pump." In non-condensing engines the steam escapes into the open air. In the latter case the temperature of the "refrigerator" must be considered at

least that of saturated steam at the atmospheric pressure, or about 100°, and the temperature of the source must be taken as that of saturated steam at the boiler pressure. Applying the expression for the efficiency (§ 160),

$$e = \frac{t - \theta}{t},$$

it will be seen, that, for any boiler pressure that it is safe to employ in practice, it is not possible, even with a perfect engine, to convert into work more than about fifteen per cent of the heat used.

In the condensing engine the temperature of the refrigerator may be taken as that of saturated steam at the pressure which exists in the condenser, say, 30° to 40°: hence $t - \theta$ is a much larger quantity for condensing than for non-condensing engines. The gain of efficiency is not, however, so great as would appear from the formula, because of the energy that must be expended to maintain the vacuum in the condenser.

163. ***Hot-air and Gas Engines.*** — Hot-air engines consist essentially of two cylinders of different capacities, with some arrangement for heating air in, or on its way to, the larger cylinder. In one form of the engine, an air-tight furnace forms the passage between the two cylinders, of which the smaller may be considered as a supply-pump for taking air from outside, and forcing it through the furnace into the larger cylinder, where, in consequence of its expansion by the heat, it is enabled to perform work. On the return stroke, this air is expelled into the external air, still hot, but at a lower temperature than it would have been had it not expanded and performed work. This case is exactly

analogous to that of the steam-engine, in which water is forced by a piston working in a small cylinder, into a boiler, there converted into steam, and then, acting upon a much larger piston, performs its work, and is rejected. In another form of the engine, known as the "ready motor," the air is forced into the large cylinder through a passage kept supplied with crude petroleum. The air becomes saturated with the vapor, forming a combustible mixture, which is burned in the cylinder itself.

The Stirling hot-air engine and the Rider "compression engine" are interesting as realizing an approach to Carnot's cycle.

These engines, like those described above, consist of two cylinders of different capacities, in which work air-tight pistons; but, unlike those, there are no valves communicating with the external atmosphere. Air is not taken in and rejected; but the same mass of air is alternately heated and cooled, alternately expands and contracts, actuating the piston, and performing work at the expense of a portion of the heat imparted to it.

It is of interest to study a little more in detail the cycle of operations in these two forms of engines. The larger of the two cylinders is kept constantly at a high temperature by means of a furnace, while the smaller is kept cold by the circulation of water. The cylinders communicate freely with each other. The pistons are actuated by cranks set at an angle of about 90°, so that twice during the revolution both pistons are moving for a short time in the same direction. Let us begin the study of the cycle at the instant that the small

piston reaches the top of its stroke. The large piston will then be toward the bottom of the cylinder, and descending. The small piston now descends, as well as the larger, the air in both cylinders is compressed, and there is but little transfer from one to the other. There is, therefore, comparatively little heat given up. The large piston, reaching its lowest point, begins to ascend, while the descent of the smaller continues. The air is rapidly transferred to the larger heated cylinder, and expands while taking heat from the highly heated surface. After the small piston has reached its lowest point, there is a short time during which both the pistons are rising and the air expanding, with but little transfer from one cylinder to the other, and with a relatively small absorption of heat. When the descent of the large piston begins, the small one still rising, the air is rapidly transferred to the smaller cylinder: its volume is diminished, and its heat is given up to the cold surface with which it is brought in contact. The completion of this operation brings the air back to the condition from which it started. It will be seen, that there are here four operations, which, while not presenting the simplicity of the four operations of Carnot, — since the first and third are not performed without transfer of heat, and the second and fourth not without change of temperature, — still furnish an example of work done by heat through a series of changes in the working substance, which brings it back, at the end of each revolution, to the same condition as at the beginning.

Gas-engines derive their power from the force developed by the combustion, within the cylinder, of a mixture of illuminating gas and air.

As compared with steam-engines, hot-air and gas engines use the working substance at a much higher temperature. $t - \theta$ is, therefore, greater, and the theoretical efficiency higher. There are, however, practical difficulties connected with the lubrication of the sliding surfaces under such high temperatures that have so far prevented the use of large engines of this class.

164. ***Water-Power.*** — Water flowing from a higher to a lower level furnishes energy for driving machinery. Many waterfalls are thus utilized. The energy theoretically available in a given time is the weight of the water that flows during that time multiplied by the height of the fall. If this energy is not utilized, it develops heat by friction of the water or of the material that may be transported by it. But water-power is only possible so long as the supply of water continues. For the supply of water we are dependent upon the rains; for the rains, we depend upon evaporation; and evaporation is maintained by solar heat. The energy of water-power is, therefore, transformed solar radiation.

165. ***Wind-Power.*** — A moving mass of air possesses energy equal to the mass multiplied by half the square of the velocity. This energy is available for propelling ships, for turning windmills, and for other work. Winds are due to a disturbance of atmospheric equilibrium by solar heat; and the energy of wind-power, like that of water-power, is, therefore, derived from solar energy.

166. ***Fuel.*** — By far the larger part of the energy employed by man for his purposes is derived from the combustion of wood and coal. This energy exists as the potential energy of chemical separation of oxygen from carbon and hydrogen. Now, we know that vege-

table matter is formed by the action of the solar rays through the mechanism of the leaf, and that coal is the carbon of plants that grew and decayed in a past geological age. The energy of wood and coal is, therefore, the transformed energy of solar radiations.

167. ***Animal Heat and Work.***—It is well known, that, in the animal tissues, the same chemical action takes place as in the burning of a candle. The oxygen taken into the lungs and absorbed by the blood combines with the fatty matters and starch, producing carbonic acid and water, as in combustion of the same substances elsewhere. Lavoisier was the first to attempt a measurement of the heat developed by the animal. He compared the heat developed with that due to the formation of the carbonic acid exhaled in a given time. Despretz and Dulong made similar experiments with more perfect apparatus, and found that the heat produced by the animal was about one-tenth greater than would have been produced by the formation by combustion of the carbonic acid and water exhaled.

These and similar experiments, although not taking into account all the chemical actions taking place in the body, leave no doubt that animal heat is due to atomic and molecular changes within the body.

The work performed by men and animals is also the transformed energy of food. Rumford, in 1798, saw this clearly; and he showed, in a paper of that date, that the amount of work done by a horse is much greater than would be obtained by using its food as fuel for a steam-engine.

Mayer, in 1845, held that an animal is a heat-engine, that every motion of the animal is a transformation into work of the heat developed in the tissues.

Hirn, in 1858, tried a very interesting series of experiments bearing upon this subject. In a closed box was placed a sort of treadmill, which a man could cause to revolve by stepping from step to step, as though to go up stairs. He thus performed work which could be measured by suitable apparatus outside the box. The treadwheel could also be made to revolve backward by a motor placed outside, when the man descended from step to step, and work was performed upon him.

Three distinct experiments were performed; and the amount of oxygen consumed by respiration, as well as the heat developed, was determined.

In the first experiment the man remained in repose; in the second he performed work by causing the wheel to revolve; in the third the wheel was made to revolve backward, and work performed upon him. In the second experiment, the amount of heat developed for a gram of oxygen consumed was much less, and in the third case much greater, than in the first; that is, in the second case, a portion of the chemical action, indicated by the absorption of oxygen, went to perform the work, and hence the failure to develop the due amount of heat, while in the third case the motor causing the treadwheel to revolve, performed work, which produced heat in addition to that due to the chemical action.

It has been thought that muscular energy is due to the waste of the muscles themselves: but experiments show that the waste of nitrogenized material is far too small in amount to account for the energy developed by the animal; and we must, therefore, conclude that the principal source of muscular energy is the oxidation of the non-nitrogenized material of the blood by the oxygen absorbed in respiration.

An animal is, then, a machine for converting the potential energy of food into mechanical work: but he is not, as Mayer supposed, a heat-engine; for he performs far more work than could be performed by a perfect heat-engine working between the same limits of temperature, and using the food as fuel.

The food of animals is of vegetable origin, and owes its energy to the solar rays. The energy of men and animals is, therefore, the transformed energy of the sun.

168. ***Ocean Currents.***—These are maintained by solar heat.

169. ***Tidal Energy.***—The tides are mainly caused by the attraction of the moon upon the waters of the earth. If the earth did not revolve upon its axis, or, rather, if it always presented one face to the moon, the elevated waters would remain stationary upon its surface, and furnish no source of energy. But as the earth revolves, the crest of the tidal wave moves apparently in the opposite direction, meets the shores of the continents, and forces the water up the bays and rivers, where energy is wasted in friction upon the shores, or may be made use of for turning mill-wheels. It is evident that all the energy derived from the tides comes from the rotation of the earth upon its axis; and a part of the energy of the earth's rotation is, therefore, being dissipated in the heat of friction they cause.

170. ***Internal Heat of the Earth.***—This and a few other forms of energy, such as that of native sulphur, iron, etc., are of little consequence as sources of useful energy. They may be considered as the remnants of the original energy of the earth.

171. ***The Sun.***—It has been seen, that, except the

energy of the tides, the sun's rays are the source of all the forms of energy practically available. It has been estimated that the heat received by the earth from the sun each year would melt a layer of ice over the entire globe a hundred feet in thickness. This represents energy equal to one-horse power for each fifty square feet of surface. The heat which reaches the earth is only $\frac{1}{2200000000}$ of the heat that leaves the sun; and the question naturally arises, What is the origin of this enormous store of energy? Helmholtz and Sir William Thomson have shown that the nebular hypothesis, which supposes the solar system to have originally existed as a chaotic mass of widely separated gravitating particles, presents to us an adequate source for all the energy of the system.

As these particles rush together by their mutual attractions, heat would be generated; and, as the condensation continues, the result would be highly heated suns and worlds.

172. ***Dissipation of Energy.***—It has been seen that only a fraction of the energy of heat is available for transformation into other forms of energy, and that such transformation is possible only when a difference of temperature exists. Every conversion of other forms of energy into heat puts it in a form from which it can be only partially recovered. Every transfer of heat from one body to another, or from one part to another of the same body, tends to equalize temperatures, and diminish the proportion of energy available for transformation. Such transfers of heat are continually taking place; and, so far as our present knowledge goes, there is a tendency toward an equality of temperature, or, in other words,

a uniform molecular motion, throughout the universe. If this condition of things were reached, although the total amount of energy existing in the universe would remain unchanged, the possibility of transformation would be at an end, and all activity and change would cease. This is the doctrine of the dissipation of energy to which our limited knowledge of the operations of nature leads us; but it must be remembered that our knowledge is very limited, and that there may be in nature the means of restoring the differences upon which all activity depends.